On the extinction of radiation by a homogeneous but spatially correlated random medium: reply to comment

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In response to comments by Borovoi [J. Opt. Soc. Am. A **19**, 2517 (2002)] on my earlier work [J. Opt. Soc. Am. A **18**, 1929 (2001)], the kinetic approach to extinction is compared with the traditional radiative transfer formalism and advantages of the former are illustrated with concrete examples. It is pointed out that the basic differential equation $dI(l) = -c \sigma I(l) dl$ already implies perfect randomness (absence of correlations) on small scales. One of the consequences is that the extinction of radiation in a negatively correlated random medium cannot be treated within the traditional framework. This limits the usefulness of the Jensen inequality. Also, simple counterexamples to theorems given in the first reference above and in Dokl. Akad. Nauk SSSR, **276**, 1374 (1984) are presented. © 2002 Optical Society of America

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I welcome the comments¹ on my recent work² and the opportunity to discuss the topic in more depth. Also, I am pleased to see that the author of Ref. 1 was able to use the formalism he developed earlier³ to rederive some of the results in Ref. 2.⁴ The approach of Ref. 1 is based on the continuous description of a random medium and uses tools such as the Jensen inequality (which was mentioned in the very first paragraph of Ref. 2 along with the references to prior work).^{7-9,10} It should be noted that the algebraic extinction of Ref. 2 is derived in Ref. 1 in a particularly simple manner by means of Laplace transform (see also Ref. 7, pp. 429-432 for practical difficulties with that approach). However, the approach developed in Ref. 2 is not merely another point of view and derivation of nonexponential extinction (or transmission) but is more fundamental and yields results not attainable by the radiative transfer method used in all of the references above. The limitation of the traditional approach can be traced back to the very first step of writing the basic differential equation, as we shall see shortly. The essential difference between the two approaches is also illustrated by simple counterexamples to theorems stated in Refs. 1 and 3, as is discussed next.

To that end, recall from Ref. 2 that "The examination proceeds at a fundamental level, a single particle at a time, in the spirit of classical kinetic theory. Neither specific radiative transfer formalism nor the concept of optical depth is used in our approach" (p. 1929). This is in sharp contrast to the traditional radiative transfer approach, which, from the outset, employs the notion of the optical depth $\tau = c \sigma l$, where c, σ , and l denote concentration, obstacle cross section, and propagation distance, respectively. The concentration, for example, is viewed in Ref. 1 as either a random function c(l) along the radiation path or as a random variable with the probability density function p(c). The author of Ref. 1 found the kinetic theory arguments used in Ref. 2 complicated and unreliable, referring to them as "sophistical speculations" (p. 2517). Therefore, in order to distill the essential difference between the two approaches, I will confine the discussion to the simplest examples examined by means of elementary geometry and probability. As in Ref. 2, the validity of the geometrical optics approximation and perfect absorption (no scattering) will be assumed throughout.

To begin, recall a simple and well-known argument used in Ref. 2 concerning the probability of transmission through a series of thin layers. The layers are so thin that no obstacle is in the shadow of another obstacle within the layer. The probability of transmission is then given by

$$p_{\rm tr} = (1 - \beta dx_1)(1 - \beta dx_2) \cdots (1 - \beta dx_m),$$
 (1)

where $\beta = c \sigma$, and the total probability is a product because layers are assumed entirely independent of one another. This independence leads directly to exponential decay. But what happens if the layers are not entirely independent of each other? Of special interest is the case of correlation length comparable to the width of the layer dx. Consider then this question in the language of the traditional approach, i.e., the "trivial" differential equation (terminology of Ref. 1),

$$dI(l) = -c \sigma I(l) dl, \qquad (2)$$

and note the following: The flux of parallel light rays I(l) is assumed to be a continuous function of the position l along the ray and perfectly constant everywhere in the plane perpendicular to the ray. However, the uniformity of the energy flux over the beam cross section is there only at the entrance into the medium (l = 0) but not after the light has traveled a certain distance. Indeed, at nonzero penetration distance, within the beam cross section, there appear shadows cast by previously encountered particles, while other parts remain illuminated. In other words, the beam cross section is "checkered." Therefore, if we em-

ploy the notion of the beam-cross-section-averaged energy flux I(l), writing the differential equation [Eq. (2)] tacitly assumes that (i) there is no overlap of obstacles within the differential layer dl and that (ii) particles within dl are positioned in a *perfectly random* manner (they show no preference toward either the dark or the illuminated regions).

It is assumption (ii) that is central to this response. This assumption of perfectly random locations is precisely the assumption of the Poisson process with respect to the number of obstacles per volume as a random variable. Hence no correlations among obstacle positions are allowed. In other words, consecutive layers or "slices" are completely independent of each other.¹² Ordered (lattice) arrays obviously violate the requirement of complete independence, and so do many partially ordered systems.

In order to focus on the origin of the exponential attenuation-an interplay of randomness in obstacle location and the possibility of an obstacle shadow overlap—we find it instructive to begin with the simplest (albeit somewhat artificial) example. Consider a series of parallel "checkerboard" screens such as depicted in Fig. 1 (such models were used to analyze transmittance of silver halide films¹⁴). Furthermore, require that the grains belonging to different screens never overlap; that is, they can be placed randomly within a given screen as long as the spot is not in the shadow cast by any of the previous screens. Then, within unit optical depth ($\tau = c \sigma l = 1$) all incident light is extinguished. This linear extinction provides a simple counterexample to the statement in Ref. 1 that exponential extinction always results in the far-field limit. Furthermore, such strong negative correlations are not critical to the violation of exponential extinction, because the argument is really about the possibility of an eventual "perfect blockage." For example, one can place the absorbers anywhere in the checkerboard with, say, binomial probability, and still the complete extinction eventually occurs at some finite distance.

More realistic examples of spherical obstacles with possibly overlapping shadows are readily constructed, as is



Fig. 1. Simple example of a distribution that causes linear (faster than exponential) extinction. Black squares indicate perfect absorbers. The absorbers are distributed completely at random, except for the constraint that no two obstacles can be aligned in the propagation direction (thus introducing correlations by forcing new absorbers to "avoid shadows"). While this distribution is anisotropic, one can readily construct more-realistic and isotropic distributions that result in extinction rates between the linear regime (a lower limit) and the exponential regime (e.g., see Fig. 3).

illustrated in Fig. 2. The perfectly random case (Poisson process) corresponds to the middle column of the figure, and left and right columns correspond to negative and positively correlated random media, respectively. [Examples of negative correlations are sedimentation (particles slowly settling in a laminar flow¹⁵), electrostatic repulsion, and a fermion gas]. The first row is a schematic depiction of the pair-correlation function defined in Ref. 2 via

$$P(1,2) = \bar{k}^2 \mathrm{d}V_1 \mathrm{d}V_2 [1 + \eta(l)], \qquad (3)$$

where $\bar{k} dV$ is the probability of finding a particle in dV, $\eta(l)$ is the pair-correlation function, and l is the separation distance between the two elementary volumes (see Ref. 2 for more details). Thus the fact that $\eta(l_0) = -0.2$ rather than $\eta(l_0) = 0$ indicates that particles are less likely to be a distance l_0 apart than in a perfectly random distribution. The corresponding realizations of obstacle distributions are shown in the second row of Fig. 2 (see Ref. 16 for details), and "time series" of photon absorption events (as observers move along the "cloud" layer) are shown in the third row. The fourth row depicts probability distributions for each of the three kinds of media, the essential element here is the transmission probability (zero absorption events). This is the main idea of Refs. 2 and 16 (see also Fig. 2 caption).

In comparing the reasoning of Fig. 2 with the discussion in Ref. 1, it is essential to realize that all three distributions in Fig. 2 have the same concentration and obstacle characteristics. In contrast, the "trivial" differential equation in Ref. 1 implies that, at least on small scale, the medium is perfectly random (middle column). The very notion of the concentration distribution p(c) or a random function c(l) implies a wide separation of three scales: interparticle distance, the scale on which concentration is defined, and the characteristic scale over which concentration is varied (e.g., see Fig. 1.3 of Ref. 17, p. 7). Such a superposition of locally Poisson processes is often called a Poisson mixture. Thus, if the relevant length scales are indeed widely separated, the traditional approach is able to describe some of the positively correlated media (right column of Fig. 2). However, the scaleseparation requirement precludes the formalism in Refs. 1 and 3 from resolving any random medium whose correlation distance is comparable to the interparticle distance. Furthermore, the preceding observation renders the Jensen inequality inappropriate insofar as the optical depth $\tau = c \sigma l$ is involved.

In particular, the reader should note that any negatively correlated media (left, second row, Fig. 2) cannot be described as a superposition of locally Poisson processes and therefore falls outside the approach in Ref. 1 and the radiative transfer approach, in general. This is a point of fundamental importance for understanding the differences between the approaches of Ref. 2 and Ref. 1. Perhaps the easiest way to see it is from the correlationfluctuation theorem for the variance of a particle count in a volume [Eq. (9) of Ref. 2), which reads

$$(\delta K)^2 = \bar{K} + \bar{\eta}\bar{K}^2, \qquad (4)$$



Fig. 2. Schematic depiction of the chain of reasoning in Refs. 2 and 16. The left column shows that a random distribution of absorbers more uniform than Poisson (e.g., electrostatic or hydrodynamic repulsion) can yield superexponential extinction. Similarly, the right column demonstrates that subexponential extinction can occur in positively correlated (clustered) media; the middle column for perfectly random absorbers is included for reference. The first row shows a characteristic pair-correlation function that can be used to generate corresponding distributions of absorbers. The second row (adapted from Ref. 16) shows a typical thin slice from each distribution. Note that all three distributions have the same concentration and obstacle characteristics. The number of absorption-event series shown in the third row can be illustrated as follows. Imagine a "neutrino" (a particle unaffected by the absorbers) traversing a given depth of a distribution and counting the number of absorbers encountered. The numbers "reported" by different neutrinos are plotted in the third row. The fourth row is the corresponding probability distribution. The probability of (photon) transmission through the layer is the probability of encountering no absorbers. Again note that the mean number of encounters is the same for all three distributions (fixed depth and concentration), so the entire argument is about the change in variance while the mean is held fixed.

where $\bar{\eta} = V^{-1} \int_{V} \eta dV$ is the volume-averaged paircorrelation function. This is a completely general result that allows for sub-Poisson fluctuations when $\bar{\eta}$ is negative. On the other hand, a superposition of locally Poisson processes (the right column of Fig. 3), which is implicitly implied throughout the treatment in Refs. 1 and 3, results in the distribution model (see Section 4 of Ref. 2):

$$p(K) = \int_0^\infty p(K|\bar{K})p(\bar{K})d\bar{K}$$
$$= \int_0^\infty \frac{\bar{K}^K \exp(-\bar{K})}{K!} p(\bar{K})d\bar{K}, \qquad (5)$$

where the vertical bar denotes conditional probability (see Ref. 18 for more details). Here the Poisson fluctuations in the particle number ride on top of the longer-scale concentration fluctuations. Recall that for conditional random variables, variances due to independent causes simply add (e.g., Ref. 19, pp. 65–66). Thus the variance of the Poisson mixture is enhanced beyond that of a pure Poisson probability density function by the variance of concentration fluctuations [the p(c) term in Ref. 1]. This means that the second term in Eq. (4) cannot be negative. Hence any random medium that is negatively correlated on at least some length scale is excluded from this description. This is a rather stringent constraint. Indeed, all obstacles have a characteristic size ($\sim \sigma^{1/2}$), and most are impenetrable. Therefore the pair correlation $\eta(l)$ must be negative unity for scales up to $\sim \sigma^{1/2}$. Thus negative correlations are ubiquitous.²⁰

Let us now come back to the extinction of light by the three kinds of media in Fig. 2. On the basis of the arguments of the last section in Ref. 2 entitled "Super-Poissonian Model Yields Slower-than-Exponential Attenuation," one might conjecture that the sub-Poissonian



Fig. 3. Illustration of correlation effects in "shadow overlap." These are "end on" samples of transversally infinite distributions of obstacles, with the propagation direction being out of the page and toward the reader. The medium is dilute (volume fraction less than 10^{-4} in all three media, but the dots in the upper "thinslice" panel are larger than actual size for better visibility). Left to right, we have a negatively correlated distribution, a perfectly random distribution, and a positively correlated (clustered) distribution. The bottom panels are the entire distributions (318 of the respective upper-panel slices). Despite seemingly minor visual differences among the three upper (thin) slices, the cumulative anti-correlated distribution is strikingly less transparent (absorber amount and obstacle cross section remaining equal for the three cases). All three absorber distributions yield intensity-versus-depth curves that begin at the same point and have equal initial slopes: then correlations set in. Therefore at least a transitional nonexponential regime is implied.

case (left column of Fig. 2) would yield faster than exponential (or enhanced exponential) attenuation. Indeed, Shaw *et al.*¹⁶ show that this is the case. Furthermore, this invalidates theorems stated in Refs. 1 and 3. For example, let us quote theorem B of Ref. 1. p. 2518; Ref. 3, p. 1375.

"A layer having a finite longitudinal size but infinite transversality exhibits lowest longitudinal transmittance in the case of the uniform distribution of scatterers inside this layer. Any redistribution of the scatterers in space forming transversal inhomogeneities will result in an increase of the longitudinal transmittance of the layer."

In the above quote the word "uniform" really refers to the middle column of Fig. 2, and the theorem applies to most situations involving a change from the middle column to the right one. However, as shown in Ref. 16 and discussed above (see also Fig. 3), redistribution toward a negatively correlated medium yields a more rapid extinction. Let us attempt a simple geometrical explanation.

Imagine light rays striking slices of equal cross section and width and populated with equal numbers of obstacles (second row of Fig. 2) and compare the three cases. The negatively correlated medium has less of an overlap between obstacle shadows belonging to adjacent screens because of, say, mutual repulsion (one way of attaining negative spatial correlations). Hence one expects a more efficient absorption of light than that of the purely random arrangement. Indeed, this is confirmed by detailed Monte Carlo simulations in Ref. 16. Here we illustrate the cumulative effects by stacking up many thin slices (Fig. 3). The effects are striking, demonstrating faster (slower) extinction in negatively (positively) correlated cases. This indicates deviations from the exponential extinction (and possibly an enhanced exponential extinction in the far field limit, with an effective extinction cross section).²¹ Figure 3 demonstrates that short-range correlations may have long-range effects.

It remains to examine the range of validity of these results (e.g., algebraic extinction) in light of the comments in Ref. 1 about the far-field limit. I think that the observation in Ref. 1 that algebraic extinction can be obtained as a Laplace transform of the concentration probability distribution p(c) is an interesting one (see also Ref. 7, p. 432). However, as already discussed, p(c) cannot describe negatively correlated media. Furthermore, the large-inhomogeneity ($a \ge L$) assumption is made in Ref. 1 to obtain the power law. However, no such assumption is needed in the approach of Ref. 2, and neither is the p(c)notion. Therefore the domain of validity is broader.

To be more precise and in keeping with the logic of Fig. 2, one expects convergence to the exponential extinction regime whenever convergence to Poisson statistics occurs. The variance equation $\overline{(\delta K)^2} = \overline{K} + \overline{\eta}\overline{K}^2$ may be used to that end. Whenever the second term becomes negligible relative to the first and the Poisson variance is reestablished, one expects the exponential regime to emerge. If the correlation length $\int_0^\infty \eta(l) dl$ is finite, then for observation distances much longer than the correlation length, one does recover the exponential regime, in agreement with the comments in Ref. 1. However, to the extent that the correlation length may be much longer than the optical depth, the intermediate regime of algebraic extinction applies to long (many optical depths) observation distances. Furthermore, the example of linear extinction in Fig. 1 illustrates the fact that the exponential regime might not occur in any limit, even if the correlation length is finite. Note also that there is a vast literature on fractal modeling of random media (e.g., see Ref. 8) in which variability on all scales is assumed so that the correlation length is infinite. Therefore the algebraic extinction results of Ref. 2 may be valid well beyond the near-field range, contrary to the assertion in Ref. 1, with the precise range of validity dependent on the functional form of the pair-correlation function. Finally, the very use of the pair-correlation function η in the last section of Ref. 2 accounts for "spatial correlations along an observation path" (p. 1932), again contrary to the assertion in Ref. 1.

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 4. I find the connection in Ref. 1 between average attenuation, cumulants, and correlation functions quite interesting, despite the use of abstract notions such as a characteristic functional (e.g., Ref. 5, pp. 63 and 405). However, the meaning of the various averages is not clearly delineated in concrete physical terms, but rather ergodicity is assumed instead. This renders the approach impractical, as one often has to deal with variability on many (sometimes all) scales, thus violating wide-sense stationarity, let alone ergodicity (e.g., whenever correlation length is comparable with the propagation distance or the medium dimensions; see Ref. 6, pp. 22–23).
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- 21. In the abstract of Ref. 1 as well as in the conclusions, the word "extinction" is apparently reserved for attenuation of a given layer with depth rather than horizontally averaged attenuation for layers that are not very deep. It seems rather pedantic to insist on replacing "extinction" with "horizontally averaged transmittance dependence on depth."