Record-Breaking Statistics for Random Walks in the Presence of Measurement Error and Noise

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We examine distance record setting by a random walker in the presence of a measurement error δ and additive noise γ and show that the mean number of (upper) records up to n steps still grows universally as $\langle R_n \rangle \sim n^{1/2}$ for large n for all jump densities, including Lévy distributions, and for all δ and γ. In contrast, the pace of record setting, measured by the amplitude of the $n^{1/2}$ growth, depends on δ and γ. In the absence of noise ($\gamma = 0$), the amplitude $S(\delta)$ is evaluated explicitly for arbitrary jump distributions and it decreases monotonically with increasing δ whereas, in the case of perfect measurement ($\delta = 0$), the corresponding amplitude $T(\gamma)$ increases with γ. The exact results for $S(\delta)$ offer a new perspective for characterizing instrumental precision by means of record counting. Our analytical results are supported by extensive numerical simulations.

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An upper record (record, for short) occurs at step n in a time series if the nth entry exceeds all previous entries. The statistics of record-breaking events in a discrete time series with independent and identically distributed (i.i.d) entries have been studied extensively [1–3]. Record statistics play a major role in time series analysis, in diverse contexts, including sports [4–7], biological evolution models [8,9], the theory of spin glasses [10,11], models of growing networks [12], analysis of climate data [13–17], and quantum chaos [18]. The quantity of central interest is the mean number of records $\langle R_n \rangle$ up to step n. For a time series with i.i.d entries, a striking universal result is that $\langle R_n \rangle \sim \ln n$ for large n [1], independent of the distribution of the individual entries. However, this universal logarithmic growth breaks down when the time series entries are strongly correlated, the simplest example being the case of a random walk where the time series represents the walker positions at discrete time steps.

While the subject of random walks has an enormous range of applications well beyond the original context of diffusion and Brownian motion, its exploration in terms of record setting is relatively recent. The basic question is as follows: how often does a random walker, moving in continuous space by jumping a random distance at each discrete time step, set a distance record, i.e., advance farther from the origin than at all prior steps? In other words, how does the mean number of such record-setting events grow with the number of steps? This is a natural question in many different contexts, such as in the evolution of stock prices [19,20] and queueing theory [21]. In the one-dimensional (1D) case, with pure diffusion, a universally valid result was found [22] for the mean of the upper record-setting events $\langle R_n \rangle$, namely, that it equals $(2/\sqrt{\pi})n^{1/2}$ for large n, where n is the number of steps, regardless of the length distribution of jumps (e.g., the result holds even for Lévy flights). This square root growth of $\langle R_n \rangle$ was also found numerically in two dimensions and three dimensions. Considering a drift, an abrupt shift in the scaling exponent from $1/2$ to 1 was identified [23]. Exact analytical results were also found in one dimension for a random walker with arbitrary drift [24,25], and for continuous time [26] and multiple [27] random walkers. In the latter case, the theoretical results agreed with an analysis of multiple stocks from the Standard & Poor's 500 index [27].

However, to apply these results to the interpretation of real experiments, the notion of a record—"advance farther from the origin than at all prior time steps"—requires closer examination. Why? Because measurement error δ and noise γ are unavoidable; for instance, δ can be the resolution of the detector while γ can describe white noise from an instrument reading. Ties become possible because of the "fuzziness," as discussed, e.g., in Refs. [17,28,29]. Hence, the following question arises: how does the presence of δ or γ affect the growth of $\langle R_n \rangle$ and the associated record-setting pace? Related questions were raised in the statistics literature, e.g., in terms of δ-exceedance records [30,31] and in the physics literature [29], but asymptotic results are available only for time series with i.i.d entries. The question has apparently never been raised in the context of correlated entries such as random walks. Does $\langle R_n \rangle \sim n^{1/2}$ scaling persist despite the presence of δ or γ and for various jump length distributions? If so, how is the amplitude of the $n^{1/2}$ growth (hereafter, "amplitude") affected? By way of preview, the universal growth exponent of $1/2$ holds but the amplitude carries the information about error and noise in distinct ways.

We define a "one-sided" record (positive maximum) so that the ith entry in a time series x_i is a random walk, record-breaking event (record, for short) if it exceeds all previous values in the sequence, i.e., if $x_i > \max(x_1, x_2, \ldots, x_{i-1})$. We henceforth interpret x_i as the distance of the random walker from the origin at the ith time step. However, because of the presence of a (fixed) δ, we define x_i to be a record (δ record) only if it exceeds all previous values in the sequence by, at least, δ. Similarly, accounting for noise, x_i is a record-breaking event if, with the addition of γ, it exceeds all previous values in the sequence. A subtlety is that in the presence of error, a record can be defined as being larger—by the amount of the error—than the last record, or than the last maximum, the two being identical in the absence of error. Here, we enumerate records larger than the previous maximum; this is more amenable to theoretical development.

We focus first on the influence of δ. Consider a discrete-time sequence $\{x_0 = 0, x_1, x_2, \ldots, \}$, representing the position of a 1D random walker starting at the origin $x_0 = 0$. The position x_m at step m is a continuous stochastic variable that evolves via the Markov rule $x_m = x_{m-1} + \eta_m$, where η_m represents the jump at step m. The η_m are i.i.d., each drawn from a symmetric and continuous jump density $f(\eta)$. Note that although the η_m's are uncorrelated, the x_m's are correlated. We are interested in the statistics of the number of records R_n up to step n. A record occurs at step m if $x_m - \delta \geq x_k$ for all $k = 0, 1, 2, \ldots, (m-1)$, where $\delta \geq 0$ represents the measurement error. For $\delta = 0$, the statistics of R_n are known to be universal, i.e., independent of the jump density $f(\eta)$ [22]; the mean record number $\langle R_n \rangle$ up to step n is [22]

$$\langle R_n \rangle = (2n+1)\binom{2n}{n} 2^{-2n} \underset{n\to\infty}{\longrightarrow} \frac{2}{\pi^{1/2}} n^{1/2}. \quad (1)$$

We now examine how $\langle R_n \rangle$ is affected by δ. We define an indicator $\sigma_m = \{1, 0\}$ with $\sigma_m = 1$ if a record occurs at step m and 0 otherwise. We call $x_0 = 0$ a record, i.e., $\sigma_0 = 1$. Then the number of records R_n up to step n is $R_n = \sum_{m=0}^{n} \sigma_m$. We average this expression over different histories. Because σ_m is a binary $\{1, 0\}$ variable, its average $\langle \sigma_m \rangle$ is just the probability that a record occurs at step m. Hence,

$$\langle R_n \rangle = \sum_{m=0}^{n} \langle \sigma_m \rangle = \sum_{m=0}^{n} r_m(\delta), \quad (2)$$

where $r_m(\delta)$ denotes the record rate, i.e., the probability that a record occurs at step m. By definition, $r_0 = 1$, and $r_m(\delta) = \mathrm{Prob}[x_m - \delta \geq \max[0, x_1, x_2, \ldots, x_{m-1}]]$. Thus, $r_m(\delta)$ is the probability of the event that the random walker, starting at the origin, reaches x_m at step m, while staying below $x_m - \delta$ at all intermediate steps between 0 and m, where one needs to finally integrate over all $x_m \geq \delta$. To compute this probability, it is convenient to change variables $y_k = x_m - x_{m-k}$, i.e., observe the sequence $\{y_k\}$ with respect to the last position and measure time backwards. Then, $r_m(\delta)$ is the probability that the new walker y_k, starting at the new origin at $k = 0$, makes a jump $\geq \delta$ at the first step and then subsequently up to m steps stays above δ, i.e., $r_m(\delta) = \mathrm{Prob}[y_1 \geq \delta, y_2 \geq \delta, \ldots, y_m \geq \delta | y_0 = 0]$.

To compute $r_m(\delta)$, we note that in the first step, the walker jumps to $y_1 = z + \delta$ from $y_0 = 0$ where $z \geq 0$ and subsequently up to $(m-1)$ steps it stays above the level δ. Writing $y_k = z_k + \delta$, we re-express $r_m(\delta)$ as

$$r_m(\delta) = \int_0^\infty f(z+\delta) q_{m-1}(z) dz, \quad (3)$$

where $q_n(z)$ is the probability that a random walker, starting initially at z, stays positive up to n steps. This persistence probability $q_n(z)$ has been thoroughly studied in the literature for random walks (see Ref. [32]) with an arbitrary jump density $f(\eta)$, and a general expression for its Laplace transform is known as the Pollaczek-Spitzer formula [33,34]. It states that

$$\int_0^\infty dz e^{-\lambda z} \sum_{n=0}^{\infty} s^n q_n(z) = \frac{1}{\lambda\sqrt{1-s}} \phi(s, \lambda), \quad (4)$$

where $\phi(s,\lambda) = \exp[-(\lambda/\pi)\int_0^\infty \ln(1 - s\hat{f}(k))/(\lambda^2 + k^2) dk]$ and $\hat{f}(k) = \int_\infty^\infty f(\eta) e^{ik\eta} d\eta$ is the Fourier transform of the jump density $f(\eta)$. Note that when $\delta \to 0$, the integral in Eq. (3) is just $q_m(0)$. Thus $r_m(0) = q_m(0)$. From Eq. (4), one can show [32] that $\sum_{m=0}^{\infty} q_m(0) s^m = 1/\sqrt{1-s}$, independent of the jump density. This is the celebrated Sparre Andersen theorem [35]; when inverted it simply gives

$$q_m(0) = \binom{2m}{m} 2^{-2m}.$$

When substituted in Eq. (2), it provides the universal result [22] in Eq. (1).

However, we are interested in $\delta > 0$. To compute $r_m(\delta)$ for large m in Eq. (3), we need the large m behavior of $q_m(z)$ for a fixed $z > 0$. This can be extracted by analyzing Eq. (4). One finds that the leading order behavior of the right side of Eq. (4) near $s = 1$ is simply $[\phi(1,\lambda)/\lambda] \times (1-s)^{-1/2}$. This means that $q_n(z)$ for large n, with fixed z, must behave like $q_n(z) \approx h(z)/\sqrt{\pi n}$. Substituting this on the left side of Eq. (4) and analyzing the leading behavior near $s = 1$ shows that the left hand side of Eq. (4) behaves as $\tilde{h}(\lambda)(1-s)^{-1/2}$, where $\tilde{h}(\lambda) = \int_0^\infty h(z) e^{-\lambda z} dz$ is the Laplace transform of $h(z)$. Comparing the left and right sides of Eq. (4), we obtain, for large n,

$$q_n(z) \approx \frac{h(z)}{\sqrt{\pi n}} \quad \text{with}$$
$$\tilde{h}(\lambda) = \int_0^\infty h(z) e^{-\lambda z} dz = \frac{1}{\lambda} \phi(1, \lambda), \quad (5)$$

where $\phi(1, \lambda)$ can be read off Eq. (4) as

$$\phi(1,\lambda) = \exp\left[-\frac{\lambda}{\pi}\int_0^\infty \frac{\ln(1-\hat{f}(k))}{\lambda^2+k^2}dk\right]. \tag{6}$$

Substituting the asymptotic behavior of $q_n(z)$ from Eq. (5) in Eq. (3), we obtain, for large m, $r_m(\delta) \approx U(\delta)/\sqrt{\pi m}$, $U(\delta) = \int_0^\infty dz f(z+\delta)h(z)$.

Finally, substituting this asymptotic behavior of $r_m(\delta)$ in Eq. (2) and summing for large n, the mean number of records is

$$\langle R_n\rangle \underset{n\to\infty}{\longrightarrow} S(\delta)n^{1/2}, \qquad S(\delta) = \frac{2}{\sqrt{\pi}}\int_0^\infty f(z+\delta)h(z)dz. \tag{7}$$

This is the main exact result: for an arbitrary jump density $f(\eta)$, the mean record number grows universally as $n^{1/2}$ for large n (as for $\delta = 0$), while the amplitude $S(\delta)$ depends nonuniversally on δ insofar as it depends explicitly on $f(\eta)$.

Although we have an exact expression for $S(\delta)$ for arbitrary $f(\eta)$, its explicit evaluation for all δ is difficult. For instance, to compute it explicitly for arbitrary jump density $f(\eta)$, we need to first compute its Fourier transform $\hat{f}(k)$, evaluate $\phi(1,\lambda)/\lambda$ from Eq. (6), then invert the Laplace transform Eq. (5) to obtain $h(z)$ and finally perform the integral in Eq. (7) to determine the amplitude $S(\delta)$.

For the special (yet ubiquitous, e.g., free paths in kinetics) case of an exponential jump density $f(\eta) = (b/2)\exp[-b|\eta|]$, it is possible to evaluate the amplitude $S(\delta)$. Here, $\hat{f}(k) = b^2/(b^2+k^2)$; substituting this in the expression of $\phi(1,\lambda)$ and integrating yields $\phi(1,\lambda) = (b+\lambda)/\lambda$. Hence, $\tilde{h}(\lambda) = (b+\lambda)/\lambda^2$. Inverting this Laplace transform gives $h(z) = 1 + bz$. Using this explicit form of $h(z)$ in the expression for $S(\delta)$ in Eq. (7) and integrating yields an exact expression for the amplitude, valid for all $\delta \geq 0$,

$$S(\delta) = \frac{2}{\sqrt{\pi}}\exp[-b\delta]. \tag{8}$$

Note that as $\delta \to 0$, one recovers the universal amplitude $2/\sqrt{\pi}$.

Consider next a jump density $f(\eta)$ whose tail decays as $f(\eta) \sim \exp[-|\eta|^a]$ for large η, where $a > 0$. Substituting this in the expression for $S(\delta)$ in Eq. (7), expanding for large δ, and using $h(0) = 1$, one can show that for large δ, $S(\delta) \sim \delta^{1-a}e^{-\delta^a}$. For example, for the Gaussian distribution, $f(\eta) = e^{-\eta^2/2\sigma^2}/\sqrt{2\pi\sigma^2}$, one finds that

$$S(\delta) \underset{\delta\to\infty}{\longrightarrow} \frac{\sqrt{2}}{\pi}\frac{\sigma}{\delta}e^{-\delta^2/2\sigma^2}. \tag{9}$$

Finally, consider jump densities with power law tails $f(\eta) \sim |\eta|^{-\mu-1}$ for large η with $\mu > 0$. For Lévy flights, $0 < \mu < 2$, whereas for jump densities with a finite variance, $\mu > 2$. In this case, rescaling $z = \delta y$ in the expression for $S(\delta)$ in Eq. (7) one gets $S(\delta) = (2/\sqrt{\pi})\delta\int_0^\infty f(\delta(y+1))h(y\delta)dy$. For large δ, the dominant contribution comes from the large argument of $h(z)$. By analyzing $\tilde{h}(\lambda)$ in Eq. (5) for large λ, we find that for large z, $h(z) \sim z^{\mu/2}$ for $\mu < 2$ and $h(z) \sim z$ for $\mu \geq 2$. Substituting this asymptotic behavior in $S(\delta)$ gives

$$S(\delta) \underset{\delta\to\infty}{\longrightarrow} \sim \delta^{-\mu+\alpha}, \tag{10}$$

where $\alpha = \mu/2$ for $\mu \leq 2$ and $\alpha = 1$ for $\mu \geq 2$. Thus, in this case $S(\delta)$ decays as a power law for large δ.

To test these analytical predictions we performed Monte Carlo simulations for the three jump densities: (i) $f(\eta) = (1/2)\exp[-|\eta|]$ (exponential, $b = 1$), (ii) $f(\eta) = (1/\sqrt{2\pi})\exp[-\eta^2/2]$ (Gaussian, $\sigma = 1$), and (iii) $f(\eta)$ drawn from a Lévy distribution with exponent $\mu = 1$, using Refs. [36–38]. While (i) and (ii) represent normal Fickian diffusion, (iii) represents non-Fickian (anomalous) diffusion, which can arise in diverse heterogeneous domains such as cells [39], cold atoms [40], and disordered porous media [41,42].

Our simulations are conducted with an ensemble of independent random walkers (5000 particles, each taking 10^6 steps), entering the 1D system at the origin, with step jump lengths drawn independently from a given probability density function (PDF). Each particle is moved from step to step according to its actual (sampled) location, without including δ; δ is added as a fixed fraction of the mean [median, for (iii)] jump length, which is chosen as unity. At each step, the particle location is calculated; the current distance value must exceed the last maximum by at least δ to qualify as a new δ record; otherwise we ignore it. The simulations confirm the $n^{1/2}$ scaling for the growth of the mean number of δ records, for all values of δ. Furthermore, the Monte Carlo simulations are compared to the three analytical predictions for $S(\delta)$ in Eqs. (8)–(10), in Fig. 1, showing excellent agreement. The amplitude $S(\delta)$ decreases from its universal value $S(0) = 2/\sqrt{\pi}$ as δ increases, so that fewer records are counted as the error increases. The decrease in $S(\delta)$ is steepest for the Gaussian PDF and has a much slower decay for the Lévy PDF, in complete agreement with theory. The slowing down in the Lévy case is due to the anomalously skewed nature of the PDF, with frequent small jumps and some enormous leaps; as a consequence, potential records set by small jumps are more prone to being eliminated by the δ error. In contrast, the Gaussian case displays a rapid decline with the increasing error, due to the compactness of the PDF, so that large jumps are rare and record events larger than the error are rarer yet.

We now examine the influence of the measurement noise γ. Let $\{x_0 = 0, x_1, x_2, \ldots, x_n\}$ represent the successive positions of the random walker. In this case, a record is registered at step m if $x_m + \mathcal{N}(0,\gamma)\Delta x > \max(0, x_0, x_1, \ldots, x_{m-1})$, where $\mathcal{N}(0,\gamma)$ is a zero-mean Gaussian random variable with standard deviation γ.

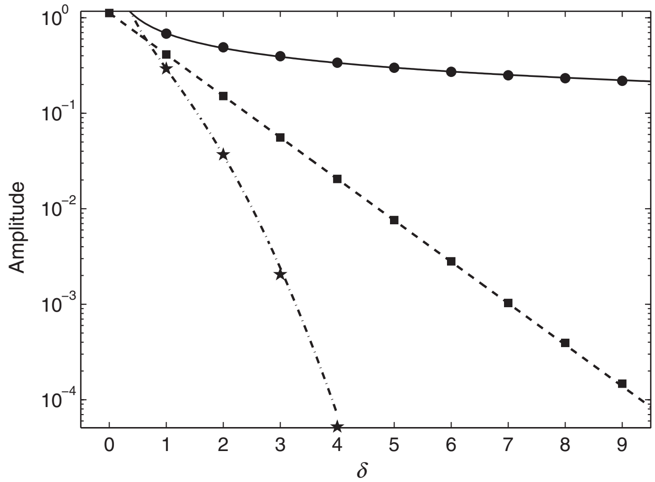

FIG. 1. One-dimensional amplitude $S(\delta)$ versus measurement error δ with Gaussian (stars), exponential (squares; $b = 1$), and Lévy (circles; $\mu = 1$) jump length PDFs. The curves (dotted dashed, dashed, solid) are the corresponding analytical results from Eqs. (9), (8), and (10) with, respectively, functional forms $(\sqrt{2}/\pi\delta)\exp[-\delta^2/2]$, $(2/\pi^{1/2})\exp(-\delta)$, and $0.69\delta^{-0.51}$. In the Lévy case, $\mu = 1$, hence $\alpha = \mu/2 = 1/2$, and the theoretical prediction $\sim\delta^{-1/2}$ in Eq. (10) is consistent with simulations.

The term $\mathcal{N}(0, \gamma)\Delta x$ mimics the measurement noise. The noise is added for the purpose of record verification at each step and is not accumulated to the actual sequence. An analytical treatment analogous to that for δ is not yet available and we resort to numerical experiments, similar to those for δ, with the results shown in Fig. 2. We use the same PDFs (i)–(iii) as before, with mean [median, for (iii)] jump length $\Delta x = 1$.

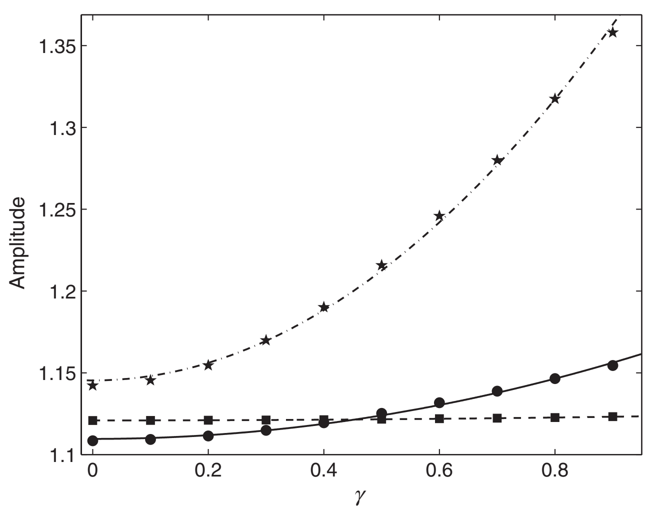

FIG. 2. Amplitude $T(\gamma)$ as a function of the measurement noise γ for jump lengths (in one dimension) with Gaussian (stars), exponential (squares; $b = 1$), and Lévy (circles; $\mu = 1$) PDFs. The curves represent quadratic fits $c_1 + c_2\gamma^2$.

While the scaling $\langle R_n\rangle \sim T(\gamma)n^{1/2}$ for large n persists, in stark contrast to the $S(\delta)$, the amplitude $T(\gamma)$ shown in Fig. 2 is an increasing function of γ for all jump densities. Thus for γ records, the noise yields false accounting of records, rendering an apparent $\langle R_n\rangle$ larger than the actual one. This spuriously large rate of record formation increases with the magnitude of the noise and suggests that it might be possible to infer the contribution of noise in diffusion-type experiments by means of record counting. One first estimates from an experiment the PDF of the jump lengths, which can then be employed in random walk simulations, to generate a curve for the amplitude $T(\gamma)$ (such as seen in Fig. 2). Returning to an ensemble of experimental measurements in the real system, one determines T and then reads off the corresponding value of γ from the simulated $T(\gamma)$ curve.

The "division of labor" discovered here, i.e., the universality of the scaling exponent, yet the contrasting dependence of the amplitude on measurement error and noise, suggests a rather different perspective on the notion of instrumental precision, among other things. To illustrate, consider implications of Eq. (8). Exponentially distributed free paths are the hallmark of kinetic theory and light scattering in random media, among others. Therefore, the instrumental precision δ of any such experiment can be inferred (in units of the mean free path $1/b$) via Eq. (8) by means of simple record counting.

The results presented here illustrate the subtlety and richness of record breaking and counting, in the presence of instrumental error δ and measurement noise γ, in systems where the underlying process can be modeled by a random walk. The decoupling of the growth exponent ($1/2$, regardless of precision and noise) from the amplitude (which depends on instrumental precision and noise in a monotonic, contrasting, and PDF-dependent manner) is significant. While the universality of the mean record number persists, $\langle R_n\rangle \sim n^{1/2}$, the magnitude of the amplitude carries the information about δ and γ.

Finally, we note that the above Monte Carlo simulations were also performed on 2D and 3D orthogonal lattices. The universality of the $n^{1/2}$ record-setting scaling is robust for all dimensions, and in all cases, the amplitudes displayed qualitative behaviors similar to those shown in Figs. 1 and 2. Moreover, Monte Carlo simulations accounting for two-sided records (absolute distance) demonstrated the same $n^{1/2}$ behavior and similar qualitative behavior for the dependence of the amplitudes on δ and γ.

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