Q. J. R. Meteorol. Soc. (2002), **128**, pp. 2045–2058

Spurious power-law relations among rainfall and radar parameters

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(Received 3 March 2001; revised 21 January 2002)

SUMMARY

In previous work, the authors examined statistical physics of rain from the point of view of the modern theory of random processes. In particular, the importance of statistical stationarity (homogeneity) was emphasized in order to attribute a clear physical meaning to the notions of drop size distribution and relations between the radar reflectivity factor, Z, and rainfall rate, R, used in radar meteorology.

In this work we return to the case of the simplest rain model, namely uncorrelated raindrops having a prescribed drop size distribution. As shown in previous work, in such rain linear relations are anticipated among the various rainfall parameters. Taking the direct approach of Monte Carlo simulations and using the techniques typical of rainfall studies over the last several decades, we then sample from a collection of rain events in which the drop occurrences are uncorrelated but each event has the same prescribed drop size distribution. Surprisingly, it is found that apparently realistic but spurious nonlinear power-law relations still appear among rainfall parameters even though the rain is not only statistically homogeneous but purely random as well.

We show that this occurs largely because of an inadequate number of drops in each sample. The drop samples typically observed, while variable in size, are often too small to sufficiently represent the size distribution or its moments. That is, each sample of rain drops, is one random, partial realization of the probability density function of diameter, that still yields functionally related pairs of variables such as Z and R. We show, however, that because of the inadequate number of drops, n, in each sample the least-square-error power-law fits yield spurious exponents that depend upon n. Thus, even though resulting fits may adequately describe the deficient observations, they are spurious and physically meaningless.

An inspection of the literature reveals that nearly all reported Z–R and other rainfall parameter relations over the last several decades are likely to be spurious because n was too small by factors of hundreds to thousands. It is clear that such fitted relations are unlikely to reflect the actual physical properties of the rain and are simply artefacts of inadequate sampling and, to a lesser extent, fitting procedures. In particular the relations of Marshall and Palmer are probably artefacts.

KEYWORDS: Drop size distributions Radar reflectivity factor Rainfall rates

1. INTRODUCTION

Along with Laws and Parsons (1943), Marshall and Palmer (1948) were among the first investigators to quantify raindrop size distributions. By performing parametric exponential regressions to their observations, they were then able to develop power-law statistical fits relating the parameters of the drop size distribution to the rainfall rate, R. Specifically, they found that

$$N\,\mathrm{d}D = N_o \exp(-\Lambda D)\,\mathrm{d}D, \tag{1}$$

while

$$\Lambda = 41 R^{-0.21}, \tag{2}$$

where $N(D)\,\mathrm{d}D$ is the number concentration for drops of diameter D to $D + \mathrm{d}D$, N_o is the intercept of the exponential fit at $D = 0$ (a constant 0.08 cm^{-4} in their analyses), Λ is the slope of the distribution (cm^{-1}), and R is the rainfall rate (mm h^{-1}).

Since the work of Marshall and Palmer (1948), hundreds of such fits have been derived in a wide variety of different types of rain and locations (for example see Battan 1973). This ever expanding accumulation of relations is now so ubiquitous that such regression fits have assumed a physical reality transcending their purely statistical nature. For example, integrating (1) from 0 to ∞ reveals that the total drop number

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concentration, n, equals N_o/Λ so that $n \propto R^{0.21}$. Consequently, it is often assumed that the number of drops in a unit volume *must* increase with increasing rainfall rate. Similarly, multiplying (1) by D and integrating, one finds that the mean drop size ($\overline{D}$) *must* increase as $R^{0.21}$ as well. Because similar calculations have been repeated so often the results now seem perfectly logical and such relations, imperative. For example, general statements such as 'the total number of drops *always* increases with increasing R', and 'the average drop size *always* increases with increasing R', are often considered to be expressions of physical necessity rather than a consequence of manipulating statistical regressions. While Jameson and Kostinski (2001a) argue extensively that such relations are not physical but rather are purely statistical in nature, we crystallize those ideas further in this paper.

This growing collection of fitted rainfall relations all essentially use the approach taken by Marshall *et al.* (1947) (for example, see Twomey 1953; Atlas 1964, Fig. 22 on page 366; Doherty 1964; Marshall 1969; Woodley and Herndon 1970; Sekhon and Srivastava 1971; Zawadzki 1975; to name but a few).We show below, however, that results such as those of Marshall and Palmer (1948) are significantly affected by the number of drops in each sample and, to a lesser extent, by the method for determining the fits.

The famous Marshall–Palmer (1948) relations are based upon 185 samples of data using dye paper exposed "...to obtain at least 100 drops per sample...collected over periods from 3–30 seconds, depending on how hard it was raining" (Marshall *et al.* 1947, page 188). Fortunately, since that time there have been significant improvements in the technology. Specifically, the large majority of past and recent regression fits have been derived using data from the Joss–Waldvogel (1967) disdrometer. This instrument has a nominal cross-sectional area of about 50 cm^2 that yields a typical drop count per sample of a few tens of drops in light rain up to a few thousands of drops in very heavy rain for the nominal one-minute sample duration. Less frequently, some regressions have been computed using aircraft measurements by the Particle Measuring Systems 2-DP probe (Knollenberg 1981). This probe has a cross-sectional area of about 17 cm^2 so that typical total counts range from tens to several thousands of drops in very heavy rain, depending on the length of the sample flight path. More recent developments in ground-based instruments such as the video disdrometer (see, for example, Schönhuber *et al.* 1997) have doubled the cross-section of the sampling area, while recent advances in aircraft probes such as the High Volume Precipitation Spectrometer (see Lawson *et al.* 1993) have increased the sampling volume a remarkable 28-fold beyond that of the 2-DP probe. Nevertheless, as impressive as these improvements are, the sample size is still often likely to remain inadequate. Moreover, regardless of the future, what is of more immediate importance here is that past rainfall regression relations are based upon samples having only tens to at most a few thousands of drops in occasional samples. So why is the total number of drops in a sample important?

The central reason is that in order to estimate the parameters as accurately as possible, it is necessary to have a sufficient number of rarer, larger drops. To see what this means, it helps first to review briefly the nature of a drop size distribution. Specifically, as discussed in Kostinski and Jameson (1999) and Jameson and Kostinski (2001a), it is very useful to separate drop concentrations from the distribution of drop diameters by rewriting (1) as

$$\begin{aligned} N(D)\,\mathrm{d}D &= n\left\{\frac{1}{\overline{D}}\exp\left(-\frac{D}{\overline{D}}\right)\right\}\,\mathrm{d}D \\ &= n \times p(D)\,\mathrm{d}D, \end{aligned} \tag{3}$$

where n is the total number of drops in a unit sample volume, $\overline{D}$ is the mean diameter and $p(D)$ is the probability density function (pdf) of the diameter. That is, $p(D)\,\mathrm{d}D$ is the probability that a drop selected at random has a diameter between D and $D + \mathrm{d}D$. (As an aside, it is shown in the above references that $N_o = n/\overline{D}$ while $\Lambda = 1/\overline{D}$. Note that n is not identical to N_o.)

Suppose we want to estimate $\overline{D}$ using n drops drawn at random from $p(D)$. Because $p(D)$ is a decreasing function of drop diameter, the probability that a particular drop drawn at random lies between D and $D + \mathrm{d}D$ is much greater for smaller drop sizes than for larger drops. It follows, then, that when n is very small, few if any large drops will be included in a limited random sample of drop diameters. However, as n increases it becomes more and more likely that larger drops will be included in a particular sample. This is particularly important for higher moments of $p(D)$.

For example, since the estimator of $\overline{D}$ is the sum of the sampled D divided by n, the estimates of $\overline{D}$ are gamma distributed when $p(D)$ is an exponential. When n is small the distribution will be very broad, but in statistically homogeneous rain in which the appearance of drops is completely random (Poissonian rain) as n increases, the distribution of $\overline{D}$ approaches a Gaussian pdf as required by the Central Limit Theorem. Although the precise functional form for higher moments corresponding to other variables such as Z and R are not as easy to specify analytically, the same statements apply to distributions of their observed values as well. Thus, the breadth of the distribution of the observed values decreases as the inverse square-root of n in such rain. This is important because most parameters are represented by different moments of $p(D)$ so that they are functionally related to each other for each observation. When n is too small, though, a range of values appears which can then often be fitted to a power-law if desired. Less important, but still a factor for some variables, the breadth of the distributions of observed values also influences the logarithmic least-square error (LSE) fitting that has been used almost exclusively in this type of research (see the appendix).

In order to explore all of these effects on past rainfall parameter regression relations, we turn to Monte Carlo experiments as described in the next section.

2. Results from Monte Carlo experiments

(*a*) *The setting*

We first consider a collection of rain events. We then assume that the rain for each event is statistically homogeneous and that the drops are uncorrelated (uncorrelated rain). This is the simplest rain, not only because there is a steady or 'constant' drop size distribution, but also because the measurements converge most rapidly towards the expected mean values since all the samples are statistically independent (see Kostinski and Jameson 1999, pages 114–115; Kostinski and Jameson 2000, appendix B; Jameson and Kostinski 2002). In such rain, the variance (breadth) of the pdf of the estimates of the mean of some parameter depends inversely upon the number of drops in a sample. (In the more general case of correlated rain, there may be a similar but much weaker dependence on sample size as well, but not always. Sometimes, the addition of more drops may not only fail to narrow the pdf, it may actually widen it! (Kostinski and Jameson 2000).) We then take samples from this collection of rain events. The famous Marshall–Palmer (1948) relations are based upon 185 samples of at least 100 drops each. As discussed above, more recent measurements may contain several hundreds to a few thousands in occasional samples. Yet, we shall show, even these increases in sample sizes appear to be wholly inadequate even in the statistically simple case of a collection of events of uncorrelated rain.

Below, we use Monte Carlo techniques to generate raindrop samples from the collection of statistically homogeneous, uncorrelated rain events in order to study what happens when samples, having different numbers of drops, are combined into an ensemble. We then perform logarithmic and linear LSE regressions to look for power-law relations. Remember, we do not have any interest here in finding the 'correct' relations; we know what they are already. Rather, our intent in this study is to perform experiments imitating what has been done for the last 60 years in the analyses of rain measurements.

Next, we imagine a fixed unit sample volume with total number of drops n evenly spread between 5 and 100. (In later calculations these numbers are increased up to 1000-fold.) These drop numbers are representative of those found in 100 litres as determined by video disdrometer observations (1-second counts), described in Jameson and Kostinski (2001a) and elsewhere, for a rain event in which R varied from a few to nearly 70 mm h^{-1}. These numbers, although smaller in total, are still quite consistent with the 1-minute counts (usually tens to a few hundreds or, occasionally, thousands, in very heavy rain) reported in past studies for the Joss–Waldvogel ground-based disdrometer. In fact, in such studies all the data, regardless of the number of drops, are often included. It is only the most astute of observers (e.g. Tokay and Short 1996, page 357) who try to avoid the sampling problems by excluding "the 1 min samples having fewer than 10 drops, or rainfall rates less than 0.1 mm h^{-1}".

We then interpret each n as the total number of draws in one realization of a sequence of random selections from $p(D)$ using an appropriate random number generator and restricting the maximum drop size to be $\overline{D} = 0.6$ cm. (This limit is now almost axiomatic in cloud physics because of the hydrodynamic-instability work of Komabayasi *et al.* (1964). Exceptions apparently do occur, however, in some tropical clouds (Beard *et al.* 1986) and when the drops form from melting ice (Blanchard 1950; Chong and Chen 1974).)

Since the rain is statistically homogeneous, we fix $p(D)$ to be a decreasing exponential function as in (3) having $\overline{D} = 0.06$ cm consistent with observations (e.g. see Fig. 15 in Jameson and Kostinski (2000)). The process is then repeated to produce 10 000 samples (realizations) to form the ensemble. For each such realization in this ensemble, we compute the sample estimate of the rainfall rate, $\widehat{R}$, the radar reflectivity factor, $\widehat{Z}$, and $\overline{D}$, where $\widehat{R}$ and $\widehat{Z}$ are given by

$$\widehat{R} = const \times \sum_{i=1}^{n} D_i^3 V_t, \tag{4a}$$

and

$$\widehat{Z} = \sum_{i=1}^{n} D_i^6, \tag{4b}$$

while D_i is the ith drop, V_t is the terminal velocity corresponding to D_i (often approximated as being $\propto D^{0.67}$ although a more accurate fit to the observations of Gunn and Kinzer (1949) is used here), and the summation is over a unit sample volume.

(*b*) *Results of Monte Carlo experiments in statistically homogeneous, uncorrelated rain*

First, let us consider how $\widehat{R}$ and $\widehat{Z}$ vary with n. Here we point out that in past work such a dependence was never even considered because drop size distributions were never written in the form given by (3). (Remember, N_o is not the same as n.)

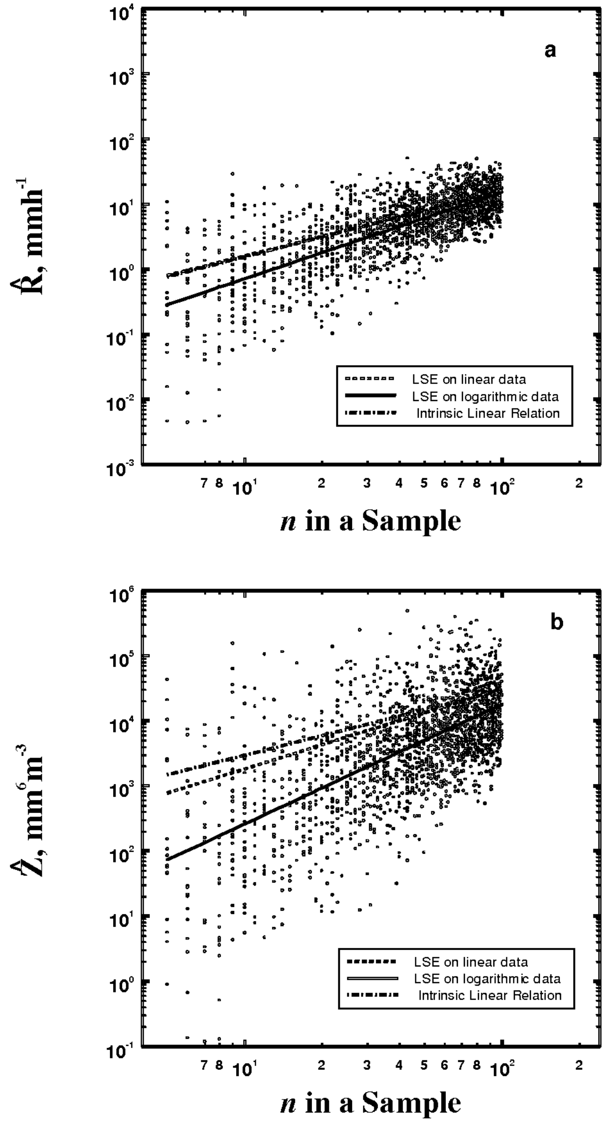

Figure 1. The linear and logarithmic least-square-error (LSE) fits between the total number of drops, n, in a unit sample volume and (a) the estimate of the rainfall rate, $\widehat{R}$, and (b) the estimate of the radar reflectivity factor, $\widehat{Z}$, for the ensemble of Monte Carlo drop samples in statistically homogeneous rain as described in the text. Note that only every fifth point is plotted. Also note the significant apparent bias in the logarithmic LSE fits.

Only when one expresses the drop size distribution in that form does the possibility of a linear dependence of $\widehat{R}$ and $\widehat{Z}$ on n emerge.

Indeed, in adequately sampled statistically homogeneous rain, Jameson and Kostinski (2001a) show that statistical regression fits between estimates of variables $\widehat{Z}$ and $\widehat{R}$ are physically based, and that each depends on n linearly. Figure 1 illustrates results from the Monte Carlo calculations. The intrinsic linear relation as well as both the linear and logarithmic LSE power-law fits between $\widehat{Z}$ and $\widehat{R}$ are plotted in Fig. 1. In the case of the former $\widehat{R} \propto n^{1.02}$, while for the logarithmic fit $\widehat{R} \propto n^{1.33}$. Similarly, the linear

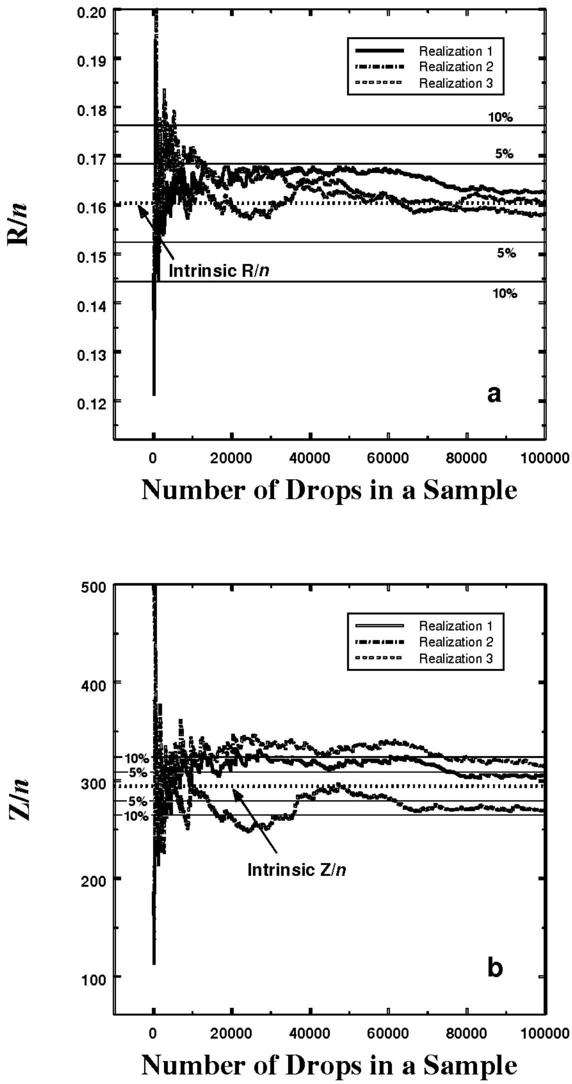

Figure 2. The ratios of (a) the rainfall rate, R, and (b) the radar reflectivity factor, Z, to the total number of drops, n, in a sample as a function of sample size for the Monte Carlo experiment described in the text. Note the large sample sizes required to reach the intrinsic linearity particularly for Z.

LSE power-law for the radar reflectivity factor goes as $\widehat{Z} \propto n^{1.23}$, while the logarithmic fit yields $\widehat{Z} \propto n^{1.86}$. Hence, for $\widehat{R}$, the power-law fit in the linear domain is essentially correct while the logarithmic power-law is biased. However, for $\widehat{Z}$ even the linear LSE power-law fit is biased. This occurs because the effect of inadequate sampling of the drop size distribution is more pronounced for higher moments of $p(D)$ as illustrated later in Figs. 3 and 4.

It could be argued that we might be better off trying to estimate $p(D)$ first as Marshall and Palmer (1948) did for every one of their 185 samples of 100 drops. But how large a total number of drops, n, does it take to determine $p(D)\,\mathrm{d}D$ even under the most statistically generous assumption of Poissonian rain? The answer depends upon the diameter bin size, ΔD, and the desired accuracy, α. That is, for a particular D and for

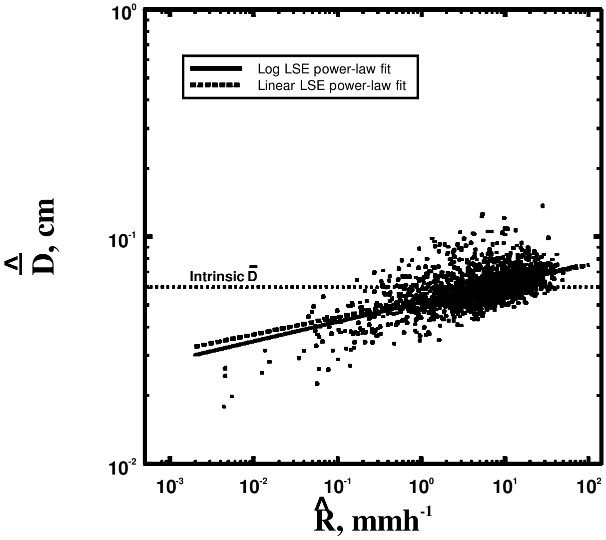

Figure 3. The least-square-error (LSE) logarithmic and linear fits between the sample mean drop diameter, $\widehat{\overline{D}}$, and the sample rainfall rate, $\widehat{R}$, for the ensemble of Monte Carlo drop samples in statistically homogeneous rain. While the intrinsic $\overline{D}$ is a constant, the logarithmic and linear LSE fits both yield spurious power laws because of inadequate sampling as discussed in the text. Note that only every fifth point is plotted.

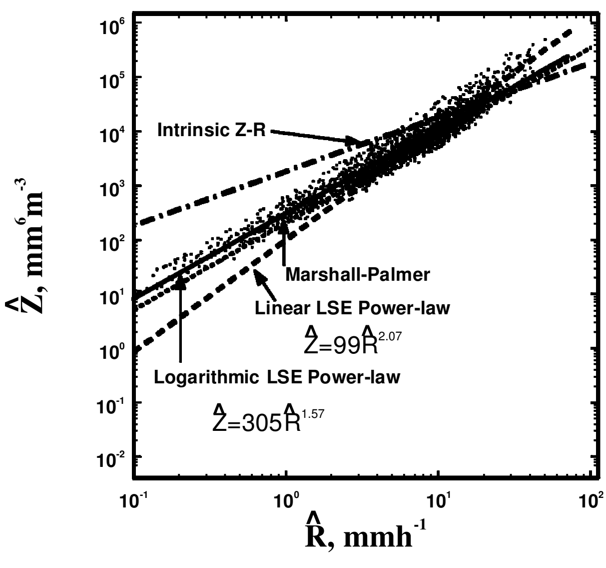

Figure 4. The linear and logarithmic least-square-error (LSE) power-law fits between the sample estimates of the radar reflectivity factor, $\widehat{Z}$, and the sample rainfall rate, $\widehat{R}$. Also plotted is the linear, intrinsic relation corresponding to statistically homogeneous rain. Note the similarity of the logarithmic fit to the relation observed by Marshall and Palmer (1948). The linear LSE power-law is also similar because the fitted data do not adequately represent the drop size distribution and these higher moments as indicated by their deviations from the intrinsic Z–R relation. As before only every fifth point is plotted. See text for further discussion.

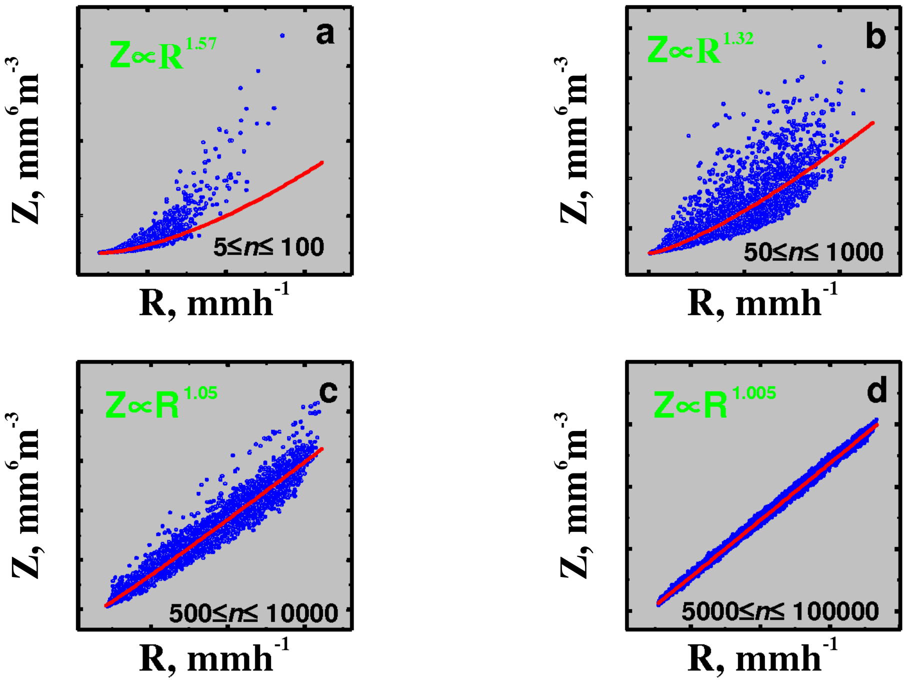

Figure 5. Logarithmic least-square-error (LSE) power-law fits to Monte Carlo simulations of 10 000 realizations from a fixed exponential probability density function of drop diameter, D, having $\overline{D} = 0.06$ cm as described in the text but with the total number of drops, n, per sample varying as indicated in each panel. Note that the exponent of the fit changes significantly depending upon n, finally converging toward the proper value of unity consistent with the intrinsic relation when n approaches values of 100 000 drops per sample. Rainfall, R, and radar reflectivity, Z, are plotted on linear scales. Also only every third point is plotted.

Poissonian rain, the number of drops in the bin ΔD around D should be $N \geq 1/\alpha^2$. This implies, in turn, that the total number of drops must be at least as large as $N/\{p(D)\Delta D)\}$ to achieve a specified accuracy. (Similar calculations have been performed by Cornford (1967) and Joss and Waldvogel (1969).) For $\Delta D = 0.02$ cm and for the $p(D)$ used in the Monte Carlo simulation, the estimation of the correct frequency of 2 mm drops to an accuracy of 10% requires a minimum sample of about 10 000 drops, while over 10^6 are required for 5 mm size drops to achieve a 10% accuracy! It makes more sense, then, to estimate the various moments of the $p(D)$ rather than $p(D)$ itself. But how many drops does that take?

To explore this, let us use our privileged perspective and first assume that we expect a linear relation between R, Z and n. We then consider the ratios R/n and Z/n plotted as functions of ever increasing sample size calculated by accumulating the individual samples from the Monte Carlo experiment. This is done for three different realizations of the experiment as illustrated in Fig. 2. Depending upon the desired accuracy and particular realization, the detection of linearity between n and R to the 10% and 5% level would require about 6000 and 13 000 drops in each sample, respectively. Similarly, 70 000 and 120 000 drops would be required for Z, respectively. Thus, while a determination of these kinds of relations requires far fewer drops than attempts to accurately specify an entire drop size distribution, the required sample sizes are still obviously considerably larger than $5 \leq n \leq 100$. Furthermore, they need to be about ten times larger for Z than for R.

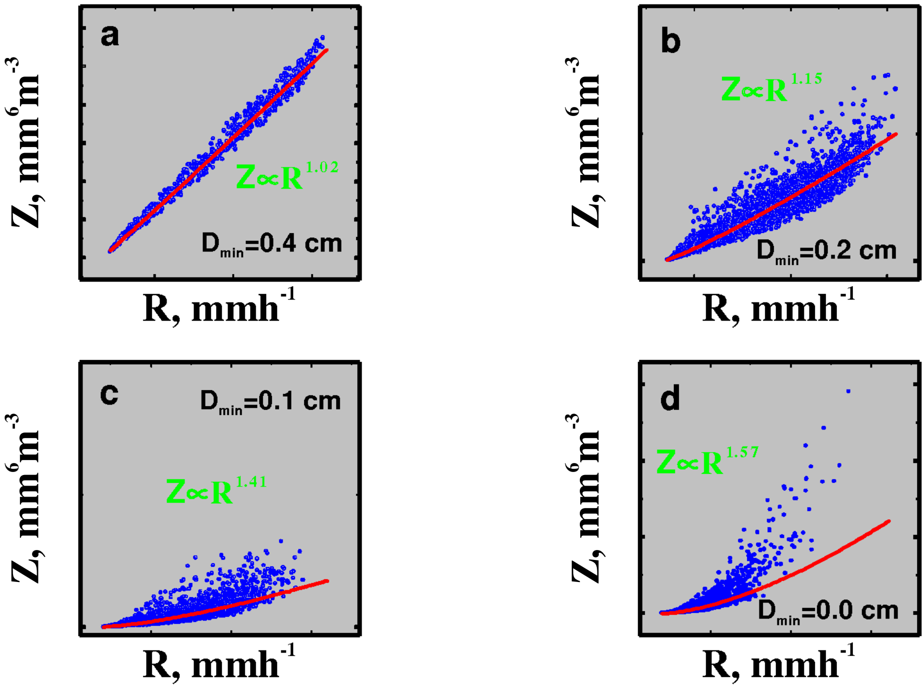

Figure 6. Similar to Fig. 5 except that now in order to explore the effect of the breadth of the drop size distribution on the exponent of the LSE logarithmic power-law fit, the total number of drops per sample is kept fixed to be $5 \leq n \leq 100$, but the range of diameters is restricted to be between the indicated minimum, D_{min}, and 0.6 cm. As expected, when the distribution is sufficiently narrow the exponent of the fit approaches that of the intrinsic relation since, even though n is small, it provides sufficient sampling of the drop size distribution. However, as the breadth of the distribution increases, the sampling over the distribution by the n drops becomes increasingly inadequate, and the exponent increasingly deviates from the correct value. Also only every third point is plotted.

Figure 1(a) suggests that the reason for biased power-law fits is simply a consequence of performing logarithmic LSE power-law fits. However, Fig. 1(b) suggests that something else is happening since even the LSE power-law fit in the linear domain exhibits some bias. This is confirmed in Fig. 3 which shows that the LSE power-laws are essentially identical whether the fit is in the linear or in the logarithmic domain even though $\overline{D}$ is a constant.

The effect is even more pronounced for fits between higher moments as illustrated in Fig. 4. Interestingly, the logarithmic LSE power-law fit is quite similar to that of Marshall and Palmer (1948) even though in our case it arises in statistically homogeneous rain rather than in their statistically inhomogeneous rain. A linear LSE power-law fit, however, also produces a similar relation with an even larger exponent reminiscent of Marshall *et al.* (1947). Thus, responsibility for the appearance of a power-law cannot be attributed to bias in the logarithmic fitting.

The generation of such artificial Z–R and similar relations in Monte Carlo simulations in past studies has been attributed to 'sampling variability', that is, to different numbers of drops in different samples (Chandrasekar and Bringi 1987; Smith *et al.* 1993). We show next, however, that the relations in those studies and in Fig. 4 are artefacts, not because of sampling variability but because of inadequate sampling; that is, there are never enough drops in any sample to measure the moments of the drop size distribution adequately, a subtle but important distinction.

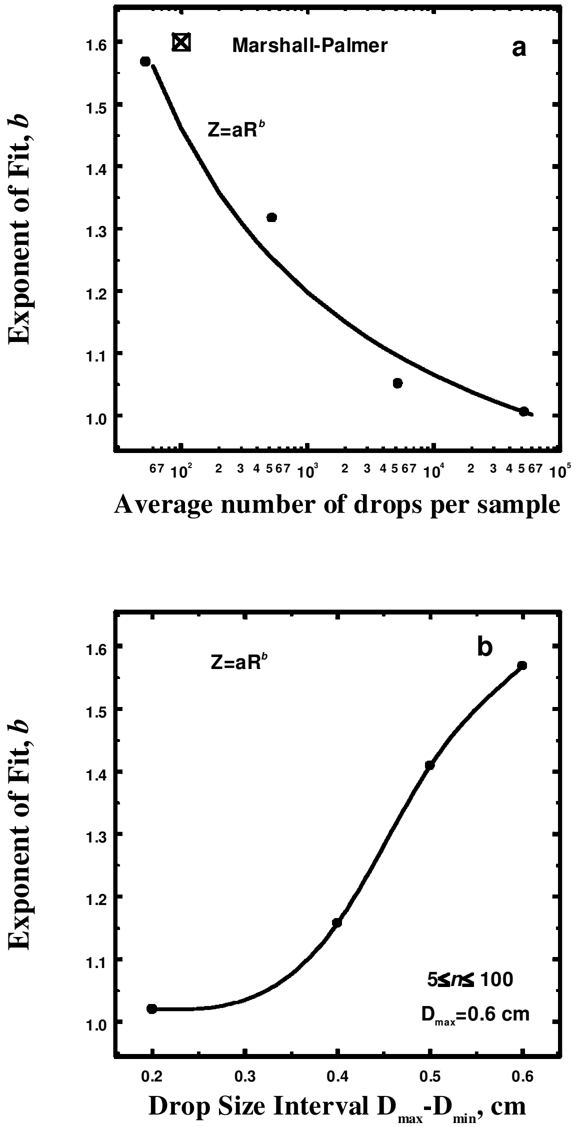

Figure 7. Spurious exponents of the logarithmic least-square-error power-law fits as functions of (a) the average total number of drops per sample and (b) as a function of the size of the diameter interval (breadth of the drop size distribution). The crossed box in (a) denotes the Marshall–Palmer (1948) observations. Note that for this particular Monte Carlo simulation, as discussed in the text, any spurious value between unity and 1.57 can be generated either by restricting the total number of drops per sample or by broadening the range of sizes over which a fixed number of drops must sample. Given the restricted number of drops per sample of most past experiments (see text), these figures indicate that the exponents reported in the literature are likely to be spurious.

This is demonstrated in Fig. 5 for the drop size distribution used in the Monte Carlo experiments. Beginning with $5 \leq n \leq 100$, we perform the identical Monte Carlo simulation of 10 000 realizations except that the number of drops in each sample is increased by factors of ten in each successive plot. The logarithmic LSE power-law fits are given in each figure. (As Fig. 4 shows, the behaviour of a linear LSE power-law fit would be similar.) Clearly, even though there is much sampling variability, we do detect the intrinsic linear relation once we have samples that each consists of several tens to hundreds of thousands of drops.

Alternatively, the effect of inadequate sampling can also be demonstrated by fixing $5 \leq n \leq 100$, but then decreasing the minimum diameter allowed (Fig. 6) while keeping the maximum diameter at 0.6 cm. In effect this forces n to sample increasingly larger domains of D. When the domain is narrow (0.2 cm), we do find the intrinsic linear relation even using a logarithmic LSE power-law fit and even when $5 \leq n \leq 100$. However, as the size of the domain increases, the exponent of the power-law fit increases as $p(D)$ is increasingly poorly sampled.

The net result is summarized in Fig. 7 for this particular drop size distribution. Clearly, depending upon n and the narrowness of the drop size distribution (domain of D), one can find a wide range of power-law exponents. In other words, all these spurious exponents are produced by inadequate sampling across $p(D)$ even though the power-laws 'fit' the observations.

3. Brief discussion

In Fig. 7 the appearance of Z–R relations like those of Marshall and Palmer (1948) in these Monte Carlo simulations is particularly sobering, because most if not all of the relations among estimates of rain parameters reported in the literature are based upon samples that are much too small, even for the statistically optimum case of homogeneous rain in which the appearance of drops is completely random. Consequently, it appears that past power-law regressions among rainfall and radar parameters, including those of Marshall *et al.* (1947) and Marshall and Palmer (1948), are very likely to be artefacts of under-sampling.

One apparently obvious solution would be to increase the number of drops in each sample. The whole idea behind large samples in statistically homogeneous rain in which the appearance of drops is completely random is to speed convergence toward the correct mean value, that is, to narrow the distribution of the estimates to acceptable limits. In this study of simulated statistically homogeneous, uncorrelated rain, such convergence occurs when the size of each sample become much, much larger than is ever measured by any instrument with the exception of radars. Unfortunately, however, such rain if it ever exists is likely to be rare.

A more likely type of rain is one that is statistically homogeneous, but in which the drops are correlated (e.g. Jameson and Kostinski 2001b, their Fig. 8). The drop correlation, however, acts to reduce significantly the effective number of independent samples (Kostinski and Jameson 1999) thereby greatly slowing any convergence to the proper mean values (narrowing of the distribution of the estimates), sometimes to such an extent that adding more drops will not help convergence at all (see Eq. 13 in Kostinski and Jameson (1999) page 114).

In most natural rain, however, the effect of drop correlation is further compounded by statistical inhomogeneity. In such rain there is no intrinsic $p(D)$ although there will be an average $p(D)$ for any particular ensemble of observations. The small sample sizes of past studies, however, are not likely to represent the actual average $p(D)$ very well (if at all). They are certainly unlikely to characterize $p(D)$ on larger scales typical, for example, of most radar measurement volumes. But even when using radar volumes containing millions of drops in each sample (e.g. Sekhon and Srivastava 1971) these regressions only apply to the particular set of statistically inhomogeneous observations used to generate them. They are not global physical laws that can be applied to other data without experimental justification. Further, since the regressions are only statistical, there is no physical justification for preferring a power-law to, say, a polynomial fit.

Fortunately, advances in radar technology mean that limited and ambiguous Z–R relations can be replaced by more direct and physically founded measurements of rain that do not depend upon such statistical regressions. Many polarization radar techniques (e.g. see discussion in Jameson (1994) and elsewhere) offer useful alternatives. If Z–R relations are to be used, however, they should be based upon very large sample sizes (at least several tens to hundreds of thousands of drops per sample), and even then they should only be applied to the data from which they are calculated.

Acknowledgements

This work was supported by the U.S. National Science Foundation under grants ATM01-0106271 (AK) and ATM00-00291 (AJ). The authors wholeheartedly thank the two anonymous reviewers for their illuminating and constructive comments.

Appendix

On biases in logarithmic least-square-error fits

Most if not all fits used in radar meteorology are not performed on the variables directly (linear LSE fitting) but rather on the logarithms of the two variables (logarithmic LSE fitting). As Fig. 1 shows, the two do not usually produce equivalent results. In part the reason for the prevalence of logarithmic fitting lies in the engineering tradition of radar meteorology that usually has to deal with a wide dynamic range of variables, so that the convenience of straight lines on log–log paper seems quite natural, particularly before computers became so accessible. However, this is also done in part because equations such as (1) and (2) erroneously (Jameson and Kostinski 2001a) seemed to provide a physical justification for computing power-law relations between Z (and other variables) and R. Consequently, these relations could, in turn, be manipulated to derive additional power-laws among other variables. It quickly became accepted that one *must* use logarithmic fits in the search for what *had* to be power-law relations among rainfall and radar parameters (Marshall 1969).

This turns out to be quite unfortunate because the breadth of the distribution of estimates of the mean values of parameters plays an important role in logarithmic LSE fits. Yet there will always be some breadth to the distribution of the estimates, even in statistically homogeneous rain in which the appearance of drops is completely random, because sample sizes are always finite. As Thompson and Macdonald (1991) point out, the "...smallest values of the original (linear) data have been unnaturally extended (weighted) relative to the larger values" when fitting using logarithms. To quote:

> "A moment of reflection will show why the biasing occurs. Consider the example of data that range from $1/M$ through unity up to M. For large M there are two sub-ranges, with lengths roughly unity and M, respectively. After taking (natural) logs, the subranges are each of length $\ln(M)$, so that the smallest values of the original data have been unnaturally extended relative to the larger values. The effect of this is to make derived parameter estimates different from the true values."

Thompson and Macdonald (1991) also show that for most error models "... the logarithmic transformation induces biases in both the exponent B and the pre-exponent A which cannot be easily corrected".

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