Complex Numbers - A review

A complex number, S, can be written using two real numbers, A and B,

$$S = A + \mathrm{i}\, B$$

where i is the square root of minus 1 ($\mathrm{i}^2 = -1$)

A is called the "Real Part" and B the "Imaginary Part" of the number.

$$\mathrm{Re}(S) = A \quad ; \qquad \mathrm{Im}(S) = B$$

Using the identity*

$$e^{ix} = \cos(x) + i \sin(x)$$

one can also write

$$S = \mathrm{C}\, e^{i\phi}$$

where C and ϕ are both real numbers,

$$\mathrm{C} = |S| = (A^2 + B^2)^{1/2}$$

That is

$$A = \mathrm{C} \cos(\phi) \; ; \quad B = \mathrm{C} \sin(\phi)$$

so

$$\tan(\phi) = B/A$$

Note that A and B look like the x and y components of a 2-dimensional vector of length $|S|$. That is, even though S is not a vector, it is often convenient to think of it as a vector with the real part corresponding to the x-component and the imaginary part corresponding to the y-component. The angle ϕ would then be the angle counter-clockwise from the x-axis.

Multiplication and addition proceed as for real numbers, except you need to include the definition of i . Hence if $S = A + iB$ and $T = C + iD$ are two complex numbers, then

$$S + T = (A+C) + i\,(B+D)$$

$$S\,T = (AC - BD) + i\,(AD + BC)$$

Note that multiplication is not the same as a dot product for vectors, however, if we write

$$\mathrm{S} = |\mathrm{S}|\, e^{i\phi} \quad ; \mathrm{T} = |\mathrm{T}|\, e^{i\theta}$$

Then

$$ST = |S|\, |T|\, e^{i(\phi+\theta)}$$

Multiplication and addition have the usual commutative and associative properties that you have for real numbers.

The complex conjugate of a complex number S is designated with an asterisk, S^* . You compute the complex conjugate by changing all "i's" to "-i's" (or changing the sign of the phase angle, ϕ). So if

$$S = A + \mathrm{i}\, B = \mathrm{C}\, e^{i\phi}, \quad \text{then } S^* = A - \mathrm{i}\, B = \mathrm{C}\, e^{-i\phi}$$

Then $|S|^2 = S\, S^*$. This is often a convenient way to find $|S|$.

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* You can prove this identity by writing out the Taylor's series expansions for the exponential and the trig functions, and using the definition of i .

Some Identities

$i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$,... , $i^{100} = (i^4)^{25} = 1$, $i^{101} = i(i^4)^{25} = i$, etc...

$e^{i\pi} = -1$, $e^{i\pi/2} = i$

$1/i = -i$ [derive from $1 = -(-1) = -(i^2)$ or similar]

$\sin(x) = (e^{ix} - e^{-ix})/2i$; $\cos(x) = (e^{ix} + e^{-ix})/2$

$e^{S+T} = e^S e^T$ where S and T are any complex numbers (including pure real and pure imaginary).

Some Problems you should be able to do:

For the following, compute without using your calculator (leave numbers exact).

1. If S and T are two complex numbers, show that $|ST| = |S|\,|T|$.

2. If $S = 3 - 4i$, what are the real and imaginary parts of $S^{-1} = 1/S$?

3. If S is a non-zero complex number, prove or disprove that it is *always* true that $|1/S| = 1/|S|$. (The more general case is $|S^n| = |S|^n$, where n is any integer).

4. If S and T are two non-zero complex numbers, prove or disprove that it is *always* true that $|S/T| = |S|/|T|$.

5. If $S = 5 + 2i$ and $T = 3 - 4i$, compute
   (A) S + T
   (B) S/T
   (C) (S-T)/(S+T)
   (D) 1/S + 1/T

(in each case, put your answer into both the form A + iB and in the form $C e^{i\phi}$)

6. If $S = 7-4i$ and $(ST - 4e^{2i}) = 6 + 2i$, what is T ?