The Exponential Family of Functions

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots$$

and using the definition $i = \sqrt{-1}$, and the series expansions above, one can easily show

$$e^x = \cosh x + \sinh x \quad ; \qquad e^{-x} = \cosh x - \sinh x$$

$$e^{ix} = \cos x + i \sin x \quad ; \qquad e^{-ix} = \cos x - i \sin x$$

$$\cosh x = (e^x + e^{-x}) / 2 \quad ; \qquad \sinh x = (e^x - e^{-x}) / 2$$

$$\cos x = (e^{ix} + e^{-ix}) / 2 \quad ; \qquad \sin x = (e^{ix} - e^{-ix}) / (2i)$$

$$\cosh(ix) = \cos x \quad ; \qquad \sinh(ix) = i \sin x$$

$$\cos(ix) = \cosh x \quad ; \qquad \sin(ix) = i \sinh x$$

One or more of these functions show up naturally as solutions, $y(x)$, to equations such as

$$\frac{dy}{dx} = a\, y \quad ; \qquad \frac{d^2y}{dx^2} = a\, y \quad ; \quad \cdots \quad ; \qquad \frac{d^ny}{dx^n} = a\, y$$

when a is a (possibly complex) constant. For example, for the harmonic oscillator, $a = -k/m$, y is displacement, and x is time in the 2nd order equation. Which particular functions you will use will depend on a and the "boundary conditions" (e.g. starting values for the harmonic oscillator, end conditions for a wave on a string, etc.) you need to satisfy.

Note: in some texts you will see the hyperbolic functions written as ch(x) and sh(x) rather than cosh(x) and sinh(x).

Exercises (for fun):

1) Show that all of the functions above are solutions to $\frac{d^4y}{dx^4} = y$. This equation arises when computing vibrations of a long thin rod.

2) Show that $y(x) = Ae^{\beta x}$ (A and β non-zero constants) always can be made to satisfy $\frac{d^ny}{dx^n} = a\, y$ for any constant $a \neq 0$ and any (positive integer) value for n, provided that an appropriate (and possibly complex) value of β is chosen.