General Length Transformations

The use of "events" and the Lorentz transformations are useful to determine a general length transformation from one inertial frame to another.

In any reference frame, the length of an object is determined by the *simultaneous* determination of the positions of opposite ends of the object. Let's suppose that in a first reference frame, one end of the object is at $x = 0$ at $t = 0$ and the other end is at $x = L$ at $t = 0$. Hence,

event 1: $x = 0, t = 0$

event 2: $x = L, t = 0$.

The distance L between these events is a proper length, since the two events occur at the same time. This is NOT to say that L is the proper length *of the object* since the object may be moving. If you imagine putting marks in the ground corresponding to the ends of the object at $t = 0$, then L is the proper length between those marks.

Now, transform those two events to another inertial reference frame moving with a speed v along the x-axis. Then $x' = \gamma(x - vt)$ and $t' = \gamma(t - xv/c^2)$ so

event 1: $x' = 0, t' = 0$

event 2: $x' = \gamma L, t' = -\gamma Lv/c^2$.

That is, an observer in the second frame says the length measurement was done incorrectly since the ends were not observed at the same time!

Now suppose the object is traveling at a speed u along the x-axis in the original frame[*] then in the second frame, the speed is

$$u' = (u-v)/(1 - uv/c^2)$$

and hence in a time $\gamma Lv/c^2$ the object will have traveled a distance $\gamma L\, u'v/c^2$. So, at $t' = 0$, one end is at $x' = 0$ (event 1) and the other is at

event 3: $x' = \gamma L + \gamma L\, u'\, v/c^2$, $t' = 0$.

The length of the object is then determined in this second frame to be

$$L' = \gamma L(1 + u'\, v/c^2) = \gamma L(1 + (uv/c^2 - v^2/c^2)/(1 - uv/c^2)\,) = L/[\gamma(1-uv/c^2)] .$$

One can check limiting cases. If the object is stationary in the second frame, then $u' = 0$, $u = v$, L' is the proper length of the object, and $L' = \gamma L$. If the object is stationary in the first frame, then $u = 0$, $u' = -v$, the proper length of the object is measured in the original frame and $L' = L/\gamma$.

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[*] One can do this for a more general motion, but here we only look at motion along x.