VISUAL DISTORTIONS NEAR A NEUTRON STAR AND BLACK HOLE

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In Press:
AMERICAN JOURNAL OF PHYSICS
1993, Vol. 61, pp. 619-631

ABSTRACT

The visual distortion effects visible to an observer traveling around and descending to the surface of an extremely compact star are described. Specifically, trips to a "normal" neutron star, a black hole, and an ultracompact neutron star with extremely high surface gravity, are described. Concepts such as multiple imaging, red- and blue-shifting, conservation of surface brightness, the photon sphere, and the existence of multiple Einstein rings are discussed in terms of what the viewer would see. Computer generated, general relativistically accurate illustrations highlighting the distortion effects are presented and discussed. A short movie (VHS) depicting many of these effects is available to those interested free of charge.

I. BACKGROUND

It is impossible for a human to travel very near a high gravity star which has a mass like that of the Sun. If, somehow, a person could survive the extremely harmful radiation that would be emitted on or near these objects, the high gravity itself would likely pose insurmountable problems. The person could not stand casually on the surface of such a star because the high surface gravity would tend to flatten them. (Lying down wouldn't help.) Were a person to orbit the star in a spaceship, however, the immense gravitational field might be overcome by a large outward centrifugal acceleration.[1] The problem in this case, however, is the extreme change in gravity between the head and toe of the person, the extreme tidal pull, would surely prove much more than annoying for any human.[2]

Nevertheless it is informative and interesting to wonder what it would look like to visit such a high gravity environment. Significant speculations on this include popular science fiction stories such as those by Forward[3] and Niven[4]. A discussion (with cartoon sketches) of a trip to a black hole appears in Kaufmann's book "The Cosmic Frontiers of General Relativity"[2]. A description of what hot spots on a high gravity neutron star would look like to an observer far away is given by Ftaclas, Kearney, and Pechenick,[5]. Other descriptions include what a typical neutron star would look like to a distant observer, including a computer drawn wire mesh diagram[6], a description of the sky as seen from the vicinity of a black hole[7−9], a description of the image of a thin accretion disk around a black hole[10], a description of how the observer would see self-images near a black hole[11], and a short computer animated movie simulating a trip around a black hole while facing the constellation Orion by Palmer and Unruh.[12] In general, however, the professional science literature has focused mainly on mathematical detail rather than observable image distortions.

In this paper the visual aspects of a journey to several different types of high gravity stars will be discussed in some detail, along with computer generated illustrations highlighting the perceived visual distortions. The three types of stars that will be discussed are a) a "normal" neutron star, b) a black hole, and c) an "ultracompact" neutron star[13] having extremely strong surface gravity. Here the speed of the traveler will always be considered small when compared to the local speed of light, so purely special relativistic effects will be ignored.

The paper is structured as follows: Section II discusses the physical principles and mathematics necessary to describe the perceived visual distortions. In Section III the types of visual distortions will be discussed generally. Section IV then proceeds to take the reader on a fantasy mission to these high gravity environments and describes what visual distortion effects the viewer would see. In Section V comments are made.

II. GRAVITATIONAL PRINCIPLES AND MATHEMATICS

The visual distortion that will be described here would be caused by gravitation in the Schwarzschild metric.[14] Einstein's general relativity[15] is not the only gravitational theory that admits the Schwarzschild metric as an exterior solution for a spherically symmetric, non-rotating gravitational field, but it is the preferred theory, and the theory that will be assumed implicitly here. The Schwarzschild metric is

$$\mathrm{ds}^2 = -(1 - R_S/r)c^2\, dt^2 + (1 - R_S/r)^{-1}\, dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2. \tag{1}$$

Here ds is a metric measure of coordinate distance r, coordinate time t and coordinate angles θ and ϕ. The term R_S, the Schwarzschild radius, refers to the radius of a black hole event horizon, and c refers to the local speed of light. R_S is directly proportional to the mass that creates the metric through $R_S = 2GM/c^2$, where G is the gravitational constant and M is the mass interior to r.

For a photon, $ds^2 = 0$. Combining this with the conservation of angular momentum allows one to express the deflection angle ϕ of a photon moving in a gravitational field[16] as

$$\Delta\phi = \int_{r_{emitted}}^{r_{observed}} \frac{dr}{r\sqrt{r^2/b^2 - 1 + R_S/r}}, \tag{2}$$

where b is a constant over the trajectory of the photon path, corresponding to a linear projected impact parameter of a photon at infinity for a photon that escapes. This impact parameter can be visualized by assuming that when the photon is far from the gravitating object it travels in a straight line; the impact parameter is the distance between the closest approach of the continuation of this straight line and the center of the gravitating object. Note that $\Delta\phi$ is not the *extra* angle deflected by the lens but the *total* change in the ϕ angle between the observer and the source, emitted at radial coordinate $r_{emitted}$ and observed at radial coordinate $r_{observed}$. This angle is measured with the lens at the vertex, and includes gravitational deflection. Therefore, for example, a source seen by an observer just over the limb of a lens which has only a small mass, and hence a negligible effect of the trajectory of the photon, has a $\Delta\phi$ near π.

An important radius is found from Eq. (2) when $\Delta\phi$ diverges to infinity. Here a photon will circle the massive star at the photon sphere. The exact location of the photon sphere is $R_P = 1.5\,R_S$. Note that a "normal" neutron star, one that has the average properties most popularly attributed to neutron stars in the current scientific literature, does not have a photon sphere. Were it somewhat more compact, however, it would have a photon sphere, and were it much more compact, it would have an event horizon and be called a black hole. For black holes and these "ultracompact" neutron stars, circular photon orbits can exist.

Photons circling at the photon sphere are not in a stable orbit[16] - any small perturbation will cause them to spiral either in or out. Photons emitted from infinity with impact parameters slightly greater than $R_B = 3\sqrt{3}R_S/2$ will spiral around the compact star near the photon sphere and then spiral out. Photons emitted from infinity with impact parameters slightly less than R_B will spiral around near the photon sphere and then spiral in, eventually colliding with the neutron star surface or falling into the black hole. It is also possible for a photon to be emitted from a ultracompact neutron star surface, orbit near the photon sphere, and then spiral back in again impacting the surface. These describe, in general, all of the distinct cases of photon orbits possible near a high gravity star. All shorter photon trajectories will lie on one of these paths.

Stated differently, the three cases of photon orbits near a gravitating body can be classified as: "always outside the photon sphere," "crossing the photon sphere," and "always inside the photon sphere." The first is the case of a photon passing the neutron star or black hole, reaching a critical radius R_c, and then escaping again toward infinity. In this case the photon does not reach or cross the photon sphere. Its distance from the star decreases monotonically until R_c, and then increases monotonically thereafter. The second case is that of a photon continuing to come toward the neutron star (or black hole) until it impacts the surface (or falls through the event horizon). Here its distance decreases monotonically. The third case is that of a photon emitted from the surface of a ultracompact neutron star, reaching a critical radius R_c, and then falling back down and again impacting the neutron star surface. This critical radius is given by the cubic equation solution[9]

$$R_c = \frac{2b}{\sqrt{3}}\cos\left[\frac{1}{3}\arccos\left(\frac{-\sqrt{27}R_S}{2b}\right) + \frac{2n\pi}{3}\right], \tag{3}$$

where $n = 0$ is for the first case and $n = 2$ is for the third case.

Photons climbing out of a gravitating object become less energetic. This loss of energy is known as a "redshifting", as photons in the visible spectrum would appear more red. Similarly, photons falling into a gravitational field become more energetic and exhibit a blueshifting. The observed energy $E_{observed}$ at radius $r_{observed}$ of a photon emitted at radius $r_{emitted}$ with energy $E_{emitted}$ is[7]

$$E_{observed} = \frac{\sqrt{1 - R_S/r_{emitted}}}{\sqrt{1 - R_S/r_{observed}}} E_{emitted}. \tag{4}$$

Note that the magnitude of the redshifting (blueshifting) effect is not a function of the emitted angle or the received angle of the photon - it depends only on how far radially the photon had to climb out of (fall into) the potential well. Also note that the power received from a continuously emitting source would have additional factors of $\sqrt{1 - R_S/r_{emitted}}/\sqrt{1 - R_S/r_{observed}}$ caused by the relative differences in the perceived rate of the number of photons emitted per unit time.

The effect a gravitational field would have on the actual perceived color of an object is more complex, however, as it depends on the distribution of photons emitted from the source at different energies relative to the sensitivity of the observer to measuring photons of different energies. For example, an object that would be described as green might be very bright in the ultra-violet - but this would not normally be perceived, as people cannot see the ultra-violet. Were this object put in a strong gravitational field and viewed from far away, so that the photons would be significantly redshifted, the strong ultra-violet emission could be shifted into violet emission and the object would look more blue, even though its light had been redshifted. This is an exceptional case, however, and redshifted objects may indeed appear more red.

III. VISUAL DISTORTION EFFECTS IN A HIGH GRAVITY ENVIRONMENT

A. Multiple images and amplification

A gravitational field may cause a single point source to appear with multiple images. For a spherical field all of these images will occur in the plane defined by the observer position, the center of the lens, and the source. These images cannot appear out of this plane because this would break the principle of conservation of angular momentum along the photon orbits. Therefore all images of a single point source will appear on a single great circle on the observer's sky.

A gravitational field may cause an extended source to appear not only multiply imaged but also greatly distorted. There is at least one feature that each of the images will maintain, however, that is the same as the original source: red- or blueshift corrected surface brightness. Any radiative process preserves the specific intensity along the beam.[17] When gravity is involved, power along the beam is not conserved, it grows or shrinks in accordance with the red- or blueshift. What is conserved can be considered to be the "corrected surface brightness" $B_c = B_r(1 - R_S/r)^2$, where B_r is the measured surface brightness at r. B_c corresponds to the surface brightness of the source measured by an observer far away from the source, and is considered to be summed over all possible energies.

For example, if an observer originally saw an unlensed circular source with constant surface brightness, a gravitational field could cause the observer to see multiple, elongated, images. Each image would have, however, the same surface brightness (B_c) as the original unlensed source, after correction for red- or blueshift factors.

The net flux that reaches the observer from any single image of the source can be either more or less than the original unlensed flux of the source. Each image will undergo an amplification A, with A not constrained to be greater than unity.[9] This means that when considered together, the images of a source seen near a large gravitational field can have more or less flux than the same source seen without the intervening gravitational field. Essentially, there are two types of amplifications a source can be seen to undergo: time distortion induced amplifications A_{time}, related directly to the slowing of time in a gravitational field that causes photons to change both their energy (red- or blueshift) and the perceived arrival rate (and hence the source's perceived power integrated over all wavelengths), and amplifications in the apparent angular size of the source, $A_{angular}$. The total amplification will be designated $A_{total} = A_{time} \text{ x } A_{angular}$. In the convention used here, all amplifications will be greater than zero.

Time induced amplifications result when the observer is at a different r from the lens center than the source. When considering only time induced amplifications, the total bolometric (incorporating all wavelengths) power received will then be changed by an amount $A_{time} = (1 - R_S/r_{emitted})^2/(1 - R_S/r_{observed})^2$.

For the sources near perfect lens - observer alignment, the angular amplification effects typically dominate over time induced amplification effects. Angular amplifications can be computed from the deflection angle Eq. (2). If a large change in angular position on the observer's sky corresponds with a small change in the angular position at the (unlensed) source location, then the source will appear to be angularly elongated and hence amplified. Similarly a source can be angularly deamplified, but this will be referred to as an angular amplification of less than unity. Angular amplification effects should be computed on the spherical sky of the observer, and so would be given by [8]

$$A_{angular} = \frac{\sin\alpha}{\sin\beta}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}\beta}\right). \tag{5}$$

Here β represents the angular distance between the lens and the source on the observer's sky in the absence of the gravitational field of the lens, while α represents this distance in the presence of the gravitational field and light deflection. The change in this angular distance, $\mathrm{d}\alpha/\mathrm{d}\beta$ can be found by application of Eq. (2). When bending angles due to gravitational effects are small, this reduces to the amplification formulae given by Refsdal[18] and Liebes.[19]

The net angular amplification of all the images of a single source can also be either more or less than the original unlensed flux of the source.[7,20] This is because the gravitational field does not change the fact that the observer still observes the same total angular area as before: 4π steradians. Therefore if the apparent angular area from some sources is greater than without the gravitational field, then there must be other sources with apparent angular area which is lower, to compensate. In practice, only relatively few images from sources that are near the observer - lens line will have angular amplifications very large ($A_{angular} >> 1$), while the rest of the sources in the sky will be slightly deamplified ($A_{angular} \lesssim 1$).

The total flux received by an observer from all the sky can again be either more or less than the original unlensed flux from all the sky.[20] A gravitational field does not create photons - it just redistributes and (red- or) blueshifts them. The observed angular redistribution and the relative time distortions, however, now act in opposite directions.

For the background sky, $A_{angular} < 1$, because now its angular area, which used to occupy the observer's entire field of view (4π steradians) in the absence of gravity, is now less by the amount of the angular size of the photon sphere of the lens. However, the photons from the sky, because of the blueshifting, are relatively more energetic and arriving relatively more often, so that $A_{time} > 1$.[7]

In other words, the background sky takes up less of the observer's sky, but the observer receives more photons per unit area, and each photon is of higher energy. Do these effects exactly cancel? No. It turns out that all observers will measure $A_{total} > 1$, with the closer the observer the greater is A_{total}.

B. Einstein rings

An important observational aspect of visual distortions in a high gravity environment that is discussed more usually in the gravitational lensing literature than in the introductory gravitation texts is called an Einstein ring.[18–19,21] Before it was shown that all images must occur in the plane defined by the observer's position, the center of the lens, and the point source. But what if these are all collinear? No plane is then defined. In this case the image of the point source would appear to the observer as an infinitesimally thin ring. This is an Einstein ring. As will be explained, numerous Einstein rings may appear simultaneously, however, and they are also important as invisible dividing lines between sets of images,[7] even when no source is distorted into a ring.

It is not generally appreciated that the mathematical formalism allows an infinite number of Einstein rings. In fact, there can be an infinite number of Einstein rings for each set of collinear observer, lens, and source points. (The infinite nature of these results breaks down as the assumptions behind the mathematical formalism become invalid.) The only Einstein ring currently discussed in the literature is the most prominent one that occurs at precise observer - lens - source alignment, and where $\Delta\phi = \pi$. Here light emitted at a specific angle from the source would be slightly deflected by the gravitational field of the lens to reach the observer. Were the source light emitted at a different angle the lens would either not be able to bend the light enough to reach the observer or too much. Since the exact observer, lens, source alignment is symmetric about the line connecting them, this source would be seen as an annular ring. This ring will be referred to as the *first* Einstein ring. (Later the term Einstein ring will be even additionally labelled by the relative radius of the source.)

Other Einstein rings can be seen angularly closer to the center of the lens. Photons from the *third* Einstein ring (the *second* Einstein ring will be defined two paragraphs below) have fully circled the lens once near the photon sphere before coming to the observer. In fact, the path of these photons crosses itself. It is possible for photons to orbit the lens an arbitrarily large number of times before coming to the observer, and each of these orbits corresponds to an Einstein ring. Therefore there are innumerable Einstein rings for this specific observer - lens - source configuration. Each Einstein ring is seen successively closer to the apparent photon sphere position. The more times the photon must circle the neutron star or black hole before reaching the observer, the more precise the direction of its emission must have been emitted to attain this trajectory, the less likely any photon will take this trajectory, the "dimmer" the Einstein ring. For this reason the higher order Einstein rings will usually carry little light when compared to the lower order Einstein rings. In fact, the relative brightness of each Einstein ring decreases exponentially.[10]

The first Einstein ring can be seen not only in a high gravity environment, but also in a low gravity environment quite a distance from much larger objects, such as normal stars, galaxies, and clusters of galaxies. In fact, complete first Einstein rings have actually been seen for radio galaxies.[22] A good review of extragalactic measurements of gravitational lens effects is given by Blandford and Narayan.[23] A good general review of low gravity gravitational lens effects is given in a book by Schneider, Ehlers, and Falco.[24]

Another set of Einstein rings is observable when the observer and source are on the same side of the lens. Then, for compact sources such as a black hole or a sufficiently compact neutron star, light from behind the observer is able to make a "U-turn" around the neutron star and come back to be visible to the observer. The Einstein ring seen from these light trajectories will be called the *second* Einstein ring, since it is seen between the first and third Einstein rings, and is brighter than the third but dimmer than the first. The fourth Einstein ring in this set is created when light does a "U-turn" near the photon sphere of the lens, then goes all the way around the lens again near the photon sphere, and finally comes to the observer. Note that there is a critical minimum (or maximum for observers inside the photon sphere) distance for the photon just like in the case of slight deflection, that is given by Eq. (3). There are also an infinite number of higher order Einstein rings of this type. As before, however, these Einstein rings carry relatively little power when compared to the lower order Einstein rings.

It is convenient to also define the zeroth Einstein ring, where light from a source located on the line from the lens through the observer comes directly undeflected to the observer along a radial line ($\Delta\phi = 0$). This Einstein "ring" is actually a single point on the observer's sky. It differs from the other Einstein rings in that its angular amplification (of a collinear point source) is not formally divergent.

Note that a single source located precisely on the opposite side of the lens from the observer would create only the first, third, fifth, etc. (i.e. odd numbered) Einstein rings. A single source located on the same side of the lens as the observer would create the zeroth, second, fourth, etc. (i.e. even numbered) Einstein rings.

In general, the position of each set of Einstein rings will be different for each specific source radius from the lens, relative to the observer position. For example, a point source at infinity directly behind the lens from the observer would create a complete set of odd numbered Einstein rings. A point source located a small, finite distance from the lens (but still directly behind the lens) would create a different set of odd numbered Einstein rings. Each set of Einstein rings can thus be labeled by the location of the source sphere. Sources at infinity will be referred to "sky" Einstein rings. For sources on the surface of the lens, the term "surface" Einstein rings will be used. In general, the convention will be taken of labelling each Einstein ring by the name or radius of the source sphere.

Mathematically, an Einstein ring will always occur when the total deflection angle due to gravitation $\Delta\phi$ (Eq. 2) is equal to any integer multiple of π radians.[7] Note that the Einstein rings are theoretical constructs and would only be visible were a source placed precisely on the observer-lens line, which for any small source is unlikely.

If the angular radius of an opaque lens is larger than the angular radius of the first Einstein ring for the source, then this ring will not exist in the sense that it will not appear on the observer's sky. If the radius of the lens is smaller than the radius of the first Einstein ring but larger than the other Einstein rings, then only the first Einstein ring will exist. If the radius of the lens is small enough so that the lens exhibits a photon sphere, however, an infinite number of Einstein rings exist. This is because a subsequent Einstein ring exists for each revolution of the lens a photon orbit can take, and theoretically, since all of these orbits are contained completely above the photon sphere, it can take an arbitrarily large number of them.

It should be noted that the existence of an Einstein ring depends on the relative positions of the lens, observer, and source, while the existence of the photon sphere or event horizon does not depend on these relative positions. It is possible for the first sky

Einstein ring to exist for a given observer looking toward a neutron star lens, but as the observer moves closer to the neutron star the angular size of the surface becomes larger than the angular size of this Einstein ring. For black holes and neutron stars compact enough to have a photon sphere, though, the photon sphere is a real entity - photons do circle there - whether or not an observer is there to see them.

C. Complete sky and surface visibility

A complete image of the sky is always contained between each two "sky" Einstein rings.[7] Likewise a complete image of the neutron star is always contained between each two "surface" Einstein rings. In general, a single complete image of all the sources on a sphere centered on the lens is visible between each two consecutive Einstein rings of that sphere.

If the radius of the lens is small enough so that the lens exhibits a photon sphere, an infinite number of images can be seen of a source, no matter its location. One image of the source comes to the observer relatively undeflected. This image is between the zeroth and first Einstein rings and will be referred to as the primary image. A second image comes around the opposite limb of the lens from the first image, and therefore will appear to the observer 180° around the face of the lens from the first image. This secondary image will always be located between the first and second Einstein rings. A third image comes around the same limb as the first image and is seen even closer to the apparent position of the photon sphere. This image has circled the neutron star or black hole fully once before reaching the observer, and its location is always between the second and third Einstein rings. The photon path for this image (and all higher order images) crosses itself. A fourth image occurs closer to but outside of the same limb as the second image, but has fully circled the lens once in the opposite direction. There is a subsequent image for each revolution of the lens a photon orbits takes, and theoretically it can take an infinite number of them. In practice, these multi-revolution images have little power and would be increasingly hard to see.[9]

Each set of images contained between successive Einstein rings is converted into "mirror writing" with respect to the images between the previous two Einstein rings. For example, if the source was a book, then the book would be visible with relatively minor distortions in its primary image - between the zeroth and first book Einstein rings. For the secondary image, between the first and second book Einstein rings, the book would appear in mirror writing, but right side up. The mapping of the entire sphere onto the annular ring between the two book Einstein rings would also cause prominent distortions. The third image of the book, between the second and third book Einstein rings, would appear in normal writing again (neither in mirror writing nor inverted), but even more distorted because of the decreased relative angular area between these two book Einstein rings. A discussion of the parity of lensed images for the brightest two images of the point lens (considered here) as well as other gravitational lens types, can be found in Blandford and Kochanek.[25]

Therefore, for a compact enough neutron star, one can see the whole neutron star surface.[5] An observer can see the complete surface of a lens (exactly once) when the first surface Einstein ring is the same angular size as the surface of the lens. (A derivation of the angular size of a sphere of mass M, radius R_* at distance D is given in the Appendix.) When the second surface Einstein ring has equal angular size to the apparent angular size of the lens surface, two complete images of the lens surface are visible.

Any lens which has a first surface Einstein ring is completely incapable of blocking light from any source. These objects cannot "eclipse" anything. This is why a neutron star in a well separated binary system can never block the light of its binary companion.

Less stringently, any lens with a first sky Einstein ring is incapable of blocking light from the background sky. Almost all stars in our galaxy are thus incapable of blocking light from random superpositions of background objects. For example, no supernovae in other galaxies are missed because they are "eclipsed" by a random superposition of a foreground star in the Milky Way Galaxy. Were such a chance superposition to occur (it is very unlikely), the supernova would be greatly amplified by the gravitational field of the intervening star rather than diminished by an "eclipse" effect. With respect to distant sources, these stars are easily compact enough to show a first Einstein ring to a distant observer, and are therefore incapable of blocking the source's light.

Every star in existence, besides the Sun but including even the nearest stars, has a first sky Einstein ring with respect to an earth bound observer. The small angular size of this Einstein ring is currently below optical resolution, but, for the nearby stars, not below the angular resolution of many radio observations. The gravity of these normal stars is strong enough to bend the background light around them and cause distant sources to be visible to the observer. Almost none of the nearby stars, however, would have a second sky Einstein ring, unless they were a neutron star or black hole. Were the star compact enough to have a photon sphere surrounding it, then, theoretically, an infinite number of sky Einstein rings (and hence sky images) could exist.

D. "Self" Einstein rings: Where to see yourself

A very interesting set of Einstein rings are the "self" Einstein rings, where observers can see themselves. The most well known of these can be seen when the observer is located at the photon sphere. There observers can simply look along the photon sphere, where light travels in a circle, and see the backs of their heads![11,26] All observers in the presence of a sufficiently compact lens, however, can see themselves. Here, light can leave the observer, travel around the lens and return to the observer to be viewed. Observers would see themselves as a series of Einstein rings. The more times light can circle the lens and return to the observer, the more "self" images the observer can see. For a lens compact enough to have a photon sphere, observers can, theoretically, see themselves in every self Einstein ring: an infinite number of times.

Amusingly, there is a single case where observers can see only a single image of themselves - and this is the case that is well known[26] - when observers are at the photon sphere! Here all the self Einstein rings actually merge with the photon sphere to form a single observer image.

Observers who see themselves would be viewing themselves with high amplification. This is because the self images observers would see would be on or near Einstein rings - which carry the highest amplifications. Therefore gravity can act as a powerful microscope! When at the photon sphere observers can microscopically view the backs of their heads, and when far away observers can microscopically view their own eyes. This is because the light that returns to the observer has left on a nearly radial trajectory - and the part of the observer most nearly radial is the observer's own eye. When close to and inside the photon sphere, observers can inspect annular rings on their heads (or spacecrafts).

IV. JOURNEY TO A HIGH GRAVITY STAR

A fairly detailed description of the distortion effects a space-traveler (or camera) would see on a visit to a high gravity star is now possible. The case that will be described first will be a trip to a "normal" neutron star: one with a currently popular equation for the interior structure of the star. This star is not dense enough to have an event horizon or a photon sphere.

The second case that will be described is that of visiting a black hole. This case is more complex in that many bound and unbound photon orbits exist near the black hole. There is, however, a somewhat simpler aspect to describing this case than the previous one in that one does not have to track surface feature distortions for a black hole.

The third and last case that will be described is that of visiting a ultracompact neutron star - one with an extreme equation of state for its interior structure that allows a mean density so high the star has a photon sphere. This is the most complicated case of all to describe as it involves all three types of photon orbits described in §II as well as requiring a description of both the sky and surface feature distortions.

To more clearly delineate what the viewer would see, a set of computer generated figures were created that document the distortion effects in terms of familiar icons. In these illustrations, the sky in the background behind the high gravity star was taken to be the night sky as viewed from present-day earth. More specifically, the background sky is taken from the Bright Star Catalogue,[27] allowing all stellar images as dim as 5th magnitude to be seen, and stellar images as dim as 7th magnitude may be amplified into visibility. In the two cases of neutron stars, a map of the earth was projected onto the surfaces of the stars and allowed to distort. These figures are, in many aspects, fully general relativistically correct. The resolution of the figures is about 3 arcminutes (0.05°).

Stellar image brightnesses are shown by the area the stellar image takes on the plots: the area is directly proportional to the flux the observer would receive from the image. It was impossible to change the pixel brightness, so many of the single pixel images would actually be seen dimmer than shown in the figures. Stellar images were allowed to get brighter or dimmer by angular amplification effects, but time induced amplification effects have been suppressed.

Note that for $A_{angular} > 1$ the stellar image flux would actually be seen as an increase in angular area of the image, so that the amplified angular area of the stellar images in the computer generated plots are, in this sense, realistic. However, the distortions in the amplified images would not be readily observable, as these background images would be unresolved by the viewer and hence indistinguishable from point sources. A small amplification would not cause the image to be resolved. Stellar images will therefore always be depicted as circles, even when they undergo angular amplification, as these convey best the idea of an unresolved point sources.

Only the two brightest images of all sources were tracked by the computer programs used. All stars originally 5th magnitude or brighter are plotted as secondary images, no matter their magnitude after gravitational distortion. Stars originally 5th magnitude are only plotted as primary images, however, if their final post-lensed magnitude was 5 or brighter. Higher order images undergoing larger angular amplification could potentially be visible but one would need significantly better angular resolution so see them (the only exception to this will be Fig. 2p), so they will be suppressed. An angular amplification limit of a factor of 100 was placed on all images for plotting purposes.

The hypothetical camera used in the simulations is somewhat fanciful but has several defining characteristics. First of all the camera is asymptotically small so that no general relativistic light bending effects are important over the length of the camera. The camera's field of view is 90^o across the middle of the picture. Lastly, the illustrations that follow have been "flat-fielded" so that angular area on the spherical sky is directly proportional to spatial area on the flat page.

A. Journey to a normal neutron star

This section will describe a trip to a fairly "normal" sized neutron star. This neutron star has a hard equation of state[28] for its internal matter, the result of which is that the matter in the star is not compressed enough to have either a photon sphere or an event horizon. The star is considered here to be non-rotating so that gravity external to its surface is described by the Schwarzschild metric and the analysis given in Section II.

It is not necessary to specify a mass of the star for a description of a trip to it. All distances can be given in terms of the Schwarzschild radius of the star, and hence are all scalable by this factor. Distances will therefore be given first in terms of the number of Schwarzschild radii, and second, in parenthesis, in terms of kilometers for a specific model. For each star, the specific model used will be for a star of mass $1.4 M_\odot$. By the simple formula $R_S = 2GM/c^2 = (3\text{ km})\,M/M_\odot$, this mass corresponds to a Schwarzschild radius of 4.2 km. The surface of this neutron star will be $R_* = 3\,R_S = 12.6$ km.

As the voyage starts, the viewer is moving through space toward the constellation Orion and the neutron star. At a distance from the neutron star of about 1000 R_S (4200 km), the neutron star destination comes into view. It is first noticeable as a very small fuzzy patch, as depicted in center of Fig. 1a. At 100 R_S (420 km) from the neutron star as depicted in Fig. 1b, the neutron star itself becomes visible and the fuzziness around it becomes resolved into a large conglomeration of individual secondary stellar images. Here for the first time the viewer can see detail inside the first sky Einstein ring. (To reiterate, Einstein rings are themselves usually invisible - they can be thought of as imaginary dividing lines between image sets.) This image conglomeration will have the same average surface brightness as the rest of the sky: if one could distribute the starlight in the rest of the sky about the whole sky it would appear to have the same apparent brightness. Although not apparent on the black and white figures, each star appears slightly blueshifted compared with its original appearance, while the surface appears more noticeably redshifted.

As the viewer approaches the neutron star it becomes evident that stars that would have appeared eclipsed behind the neutron star in the absence of its gravitational field now appear to have two bright images: one just outside the first sky Einstein ring and one just inside. This is depicted in Fig. 1c where the viewer is now 25 R_S (105 km) from the center of the neutron star, 12.4 R_S (52.08 km) from its surface.

The viewer now descends to only 10 R_S (42 km) from the star's center. Visual distortions appear as in Fig. 1d. Many of the stellar images interior to the first sky Einstein ring can be very clearly resolved. No other Einstein rings are in the field of view. At this distance from the neutron star surface features can now be clearly resolved. Surface distortions can be highlighted by comparing the figure to a standard globe of the earth. More than half of the surface is visible here - but not the entire surface. The sky appears slightly more blueshifted than before, while the neutron star surface appears slightly less redshifted.

For clarity, the first sky Einstein ring has been drawn in with a dashed line in Fig. 1d. Also labels pointing out both the primary and secondary image of the belt of Orion are shown, as well as labels pointing out both the primary and secondary image of Sirius.

The viewer has now stopped at 10 R_S (42 km) and begins a circular orbit of the star. Figs. 1e-1j show views at relative orbital angles of 5, 10, 90, 180, 270, and 360 to the original viewer position. Fig. 1j is the same as Fig. 1d but is included to provide continuity in the presented sequence. It is particularly illuminating to compare Figs. 1d, 1e, and 1f. There the viewer can most clearly see the position of the first sky Einstein ring by the effect of the viewer's slight motion.

Note that although all stars in the sky have at least one image (the primary image between the zeroth and first sky Einstein ring), not all stars have two images. This is because the neutron star is not compact enough to allow a second sky Einstein ring at the present observer location. Therefore the whole sky image between the first and second sky Einstein rings is *not* entirely visible.

Background stars that would have appeared blocked by the neutron star in the absence of high gravity now appear greatly amplified and positioned near the first sky Einstein ring. These stars now have two bright images that appear on opposite sides of the face of the neutron star. When the observer is in orbit about the neutron star, stellar images outside the first sky Einstein ring still appear to revolve (due to the viewer's motion) in the same general sense as they would in the absence of gravity. Now, however, many of them have a secondary image visible inside the first sky Einstein ring. These images, by conservation of angular momentum, must remain in the plane defined by the source, observer, and lens center, but on the opposite side of the neutron star from the first image. Therefore these images, although rotating in the same sense (clockwise or counter-clockwise) as the first images, appear to counter-rotate around the neutron star center when compared to the stellar images just across the first sky Einstein ring from them.

The hypothetical point on the sky directly opposite the observer through the lens center has been transformed into a series of circles, the most prominent of which is the frequently discussed first sky Einstein ring. A stellar image can never be seen to cross a sky Einstein ring. For example, a single stellar image cannot move from the first complete sky image to the second complete sky image. When a star approaches this point, it will either pass above or below it. If it passes above, the primary image will appear to become greatly amplified and pass above the first sky Einstein ring. If it passes below, then the primary image will appear to become greatly amplified and pass below the first sky Einstein ring. Stars that would have been seen to approach this point in the absence of strong gravity now have images that are seen to approach the circle in the presence of strong gravity.

The description of the apparent motions of surface features are quite similar to those of the sky features. The first surface Einstein ring is not in the field of view, however, and so a complete image of the surface is not in view. It *is* possible, then, for some surface features to completely disappear from view. Notice that surface features that appear near the limb of the star are somewhat distorted. Surface features will not appear dimmer near the limb. This is a consequence of the conservation of surface brightness.

The viewer now begins to land on the neutron star. Fig. 1k shows the distortions from 8 R_S (33.6 km) and a viewing angle of 15° from looking directly at the neutron star. The next figures is sequence, Figs. 1l, 1m, 1n, 1o, and 1p, show the distortions from distances of 7 R_S (29.4 km), 6 R_S (25.2 km), 5 R_S (21 km), 4 R_S (16.8 km), and 3 R_S (12.6 km) while the viewing angle pans up from the star to angles 30°, 45°, 60°, 75°, and 90° respectively.

Fig. 1p depicts the distortions a viewer would see from the neutron star surface looking directly tangent to the surface (90° from looking directly at the neutron star). From the surface one can no longer see any sky Einstein rings and hence no secondary images of background stars. From the surface, the sky appears more blueshifted than ever before, while the surface now appears without redshift. This is because photons from the sky fall toward the neutron star further than before, while photons from the surface now do not have to climb out of a gravity well to reach the viewer.

B. Journey to a black hole

This section will describe a trip to the most compact star imaginable: a black hole. A black hole can be thought of as any star compressed so greatly that it not only has a photon sphere but also an event horizon. The black hole discussed here will be considered to be non-rotating so that gravity external to its event horizon is described by the Schwarzschild metric and the analysis given in Section II.

The trip described can be to a black hole with any given mass (and hence Schwarzschild radius). For the example model given here, a black hole with a mass of 1.4 $M_\odot$ will be used. All radial distances will be given first in terms of the Schwarzschild radius and later, in parenthesis, in terms of kilometers for this specific model. One cannot simply measure this distance with a series of meter sticks, though. This is because, for one reason, any meter stick closer than the event horizon could not be seen by an observer outside the event horizon. A better way of visualizing radial distance is to picture orbiting the black hole at a fixed distance, measuring the circumference of the orbit, and dividing by 2π.

Far from the black hole an undistorted night sky is visible with a very small patch of fuzz in the center. As the viewer nears the black hole the fuzzy patch becomes discernable as an unusual conglomeration of stellar images. Fig. 2a depicts the view visible from 1000 R_S (4200 km) away.

Fig. 2b shows the black hole from a distance of 100 R_S (420 km). From this distance the viewer begins to notice that no light comes from a circular patch in the direction of the black hole. The only light that could possible come to the viewer from this area would be from the black hole itself. Since here it is considered that the black hole emits no light[29], this area is dark. The angular size of the filled black circle is the angular size of the photon sphere of the black hole mass and can be found from the discussion in the Appendix.

Fig. 2c shows the black hole from a distance of 25 R_S (105 km). Here the angular size of the black hole has increased and the secondary images, which are inside the first sky Einstein ring, are now quite clearly discernable.

Fig. 2d shows the black hole from a distance of 10 R_S (42.0 km). Here the viewer should notice that the placements of stellar images near the black hole have changed greatly when compared to Fig. 2a. Stars nearest to behind the black hole from the observer now have two bright images. The brightest star in the illustration (and the sky: Sirius) can be seen to have two bright images: the brightest primary image in the field of view on the lower left and a secondary image 180° across the face of the black hole from it. Primary and secondary images can always be matched up by connecting them with a Great Circle (a line on these figures) through the center of the black hole. Sirius is not the only star to have two distinct images, however. Notice that Betelgeuse and each star in the belt of Orion also has two bright images. Sirius and the stars in the belt of Orion have been labelled in Fig. 2d. In fact, all bright stars visible in the field have two bright images. Some dim stars that previously could not be seen now have been amplified by the gravitation of the black hole to exhibit observably bright images. In Fig. 2d, the first sky Einstein ring has been drawn in with a dashed line.

The first sky Einstein ring, shown in Fig. 2d, is an invisible circle centered on the black hole and dividing the first complete set of images (those angularly furthest from the disk of the black hole which lie between the zeroth and first Einstein rings) from the second complete set of images. Each image in the first set is always brighter than the corresponding image in the second set. The second sky Einstein ring appears in the conglomeration of stellar images near the apparent photon sphere position, just outside the photon sphere. A complete image of the sky can be seen between these two Einstein rings.

Note that typically stellar images get much dimmer as one looks closer to the apparent photon sphere position, but the average surface brightness of the sky there remains unchanged. In other words, if Fig. 2d was spun about the center of the black hole smearing all the star images into a blur, the inner regions near the apparent position of the photon sphere would appear to have the same average brightness as the outer regions near the edge of the illustration. This is a consequence of conservation of surface brightness discussed in §III A. Due to the proximity of the black hole, the sky would appear slightly blueshifted compared to the original colors, though this isn't evident on the black and white figures.

The viewer now does an orbit around the black hole at the radius of 10 R_S (42 km). The distortions the viewer would see are shown in Figs. 2d - 2j. These figures depict viewing angles for relative angular positions of 0°, 5°, 10°, 90°, 180°, 270°, and 360° around the orbit. A complete orbit would encompass, of course, 360° and so Fig. 2j is the same as Fig. 2d.

Fig. 2e, showing a relative 5° orbital angle compared to Fig. 2d, has several interesting differences with this figure. Stellar images nearest the first sky Einstein ring have shifted the most. These images represent stars that are closest to directly behind the black hole from the viewer. These images appear to move with the highest angular speeds. This is because a small angular (unlensed) step of the star from just to the left of behind the black hole from the observer to just to the right causes all of its images to move from one side of the Einstein ring to the other. Apparent angular speeds have no maximum limit. If one attributes a distance to the images they can even appear to exceed the speed of light. Note that images of the same star still appear 180° across the face of the black hole from each other, and that the brighter image is outside the first Einstein ring, while the dimmer image is inside.

Remember, an entire single image of the sky is contained between the zeroth and first sky Einstein rings. It is therefore impossible for an image to leave this region - it cannot just "go" across this ring and end up between the first and second Einstein rings. Stars (in reality) approaching the nadir point below the black hole from the viewer (moving slowly) have images that appear to approach the Einstein ring and get very bright (moving rapidly), and then subsequently recede from this Einstein ring and return to their original brightness.

Now the viewer will go even closer to the black hole. Fig. 2k shows the visible distortions from 3 R_S (12.6 km): at twice the distance of the photon sphere. Here the viewer is looking 45° away from the black hole. Note the greater number of clearly resolved secondary images visible near the black hole's limb.

The viewer now reaches the photon sphere and looks up from the black hole to peer directly along the photon sphere. Fig. 2l shows the distortions from this distance: 1.5 R_S (6.3 km). The viewer looks north. The self Einstein ring where viewers could see the backs of their heads is the photon sphere horizon line dividing the light captured by the black hole from the the light coming from the sky: it is a horizontal line across the middle of the figure. The first sky Einstein ring would be an invisible horizontal line about 2/9 of the way toward the top of the plot above the photon sphere. Since the viewer's location and orientation do not allow the whole face of the black hole to be visible, both of the bright images (the primary and secondary image) of a single star are not visible at the same time. Those stellar images highly amplified above the Einstein ring are different than those that appear highly amplified just below the Einstein ring. The primary images just above the Einstein ring in one direction will have their secondary image appear just below the Einstein ring in the opposite direction. Due to the extreme proximity of the black hole, the sky now appears with an even greater blueshift than before: each photon has about 73 % more energy than it would in the absence of the black hole's gravity

The viewer now starts along an orbit at the photon sphere, 1.5 R_S (6.3 km) from the black hole. The position of the first sky Einstein ring becomes more evident when comparing Figs. 2l, 2m, and 2n which have relative orbital angles of 5° and 10°.

The viewer now descends and looks directly away from the black hole. Fig. 2o shows the distortions from 1.1 R_S (4.62 km). All of the sky images are now compressed into a hole in the direction opposite the black hole.

Fig. 2p shows the distortions visible from 1.01 R_S (4.242 km) while looking directly away from the black hole. The black hole now encompasses almost the complete observer sky. The small hole at the top is what remains visible of the outside universe. In this hole there could appear, theoretically, an infinite number of complete images of the outside universe. The angular amplification $A_{angular}$ of the vast majority of these images is, however, much less than unity: they are greatly deamplified. Every sky image has almost exactly the same A_{time}, though, which corresponds to very strong blueshift.

Fig. 2p, as shown, is not an accurate depiction of the distortions a viewer would see from this position. It is included because part of it is correct and the part that is not is informative. The part that is correct is the depiction of the relative amounts of black hole and background sky that are visible. The thin fuzzy annular ring is not realistic, however, as the program plotted mostly just the positions of the secondary images. Only a handful of primary sky images are visible as most of them have suffered large angular deamplifications. As stated in the introduction to §IV, the secondary images are plotted by the program regardless of the amount of deamplification. The outer radial limit of the dim ring marks the position of the second sky Einstein ring. Stellar images that would be seen between there and the apparent photon sphere limb of the black hole would be even higher order images. This is the only figure where these images are noticeable by their absence. The programs were not set up to track these higher order images, and so they are not shown.

C. Journey to an ultracompact neutron star

This section will describe a trip to a very compact neutron star. This neutron star must have an extremely soft equation of state[30] for its internal matter, the result of which is that the matter in the star is compressed enough to exhibit a photon sphere but not compressed enough to exhibit an event horizon. The star, called "ultracompact,"[13] is considered here to be non-rotating, so that gravity external to its surface is described by the Schwarzschild metric and the analysis given in Section II.

The trip described can be to a ultracompact neutron star with any given mass (and hence Schwarzschild radius), and therefore, as in the previous sections, all distances will first be given in terms of the Schwarzschild radius, and later, in parenthesis, in terms of kilometers for a specific model. For the canonical specific model, a star will again be used with mass 1.4 $M_\odot$, but this time with a surface of $R_* = 4/3\, R_S$, the minimum allowed without violating the dominant energy condition.[31] More specifically, this hypothetical star has $R_S = 4.2$ km and $R_* = 5.6$ km.

Any object compact enough to have a photon sphere will always appear to have the apparent size of its photon sphere.[32] This is because photons coming to the viewer from the object's limb must have orbited near the photon sphere before escaping to be seen. Therefore, one cannot measure the actual stellar radius of such an object by measuring the angular radius seen. Were the star to shrink the angular radius would appear to be the same. One cannot even measure a decrease in R_* by noting the change in the positions of background stellar images, because there will be no change. The only change the observer could note is that of increased apparent distortion of surface features.

The hypothetical journey begins 100 R_S (420 km) from the neutron star. Fig. 3a shows the distortion effects a viewer would see from this distance. The distortions of the background sky are precisely the same as in the black hole description in the previous section. Essentially an undistorted night sky is visible from the viewer's location with a small patch of barely resolved fuzz in the center. The constellation Orion is clearly visible to the right of center.

As the viewer nears the ultracompact neutron star the fuzzy patch breaks up to become a conglomeration of surface images and secondary star images. This is shown in the succession of Figs. 3b, 3c, and 3d. Fig. 3b shows the neutron star from a distance of 50 R_S (210 km). Fig. 3c shows the neutron star from a distance of 25 R_S (105 km), while Fig. 3d shows the neutron star from a distance of 10 R_S (42 km).

There are several interesting aspects of Fig. 3d. The positions of the stellar images are exactly the same as if the lens were a black hole. The blueshifting of the image colors would also be exactly the same. The first sky Einstein ring has been drawn in, and the primary and secondary images of Sirius and the belt of Orion are labelled. The whole surface is visible, and some portions can even be seen to have a second image just inside the apparent position of the photon sphere. The first surface Einstein ring can be seen to have about 95 percent of the radius of the apparent position of the photon sphere in Fig. 3d. A complete image of the whole surface of the neutron star is visible inside this first surface Einstein ring. Another 'mirror written' and greatly distorted version of the entire surface would be seen in the annular space between the first and second surface Einstein rings (nearer the limb). Higher order sets of images are not shown but would be difficult to see as they would occupy increasingly thinner annular rings seen increasingly close to the photon sphere limb.

A surface feature near the limb of the neutron star would have the same surface brightness as an identical surface feature near the center, as seen by any observer. This is again a result of the conservation of surface brightness and that the surface emission is assumed isotropic.

The viewer now does an orbit around the neutron star at the radius of 10 R_S (42 km). The distortions the viewer would see are shown in Figs. 3d - 3j. These figures depict viewing angles for relative angular positions of 0°, 5°, 10°, 90°, 180°, 270°, and 360° around the orbit. A complete orbit would encompass, of course, 360°, and so Fig. 3j is the same as Fig. 3d.

Figs. 3e and 3f, showing a relative 5° and 10° angle compared to Fig. 3d, shows revealing differences when compared with Fig. 3d. The comparison allows the reader to discern the first sky Einstein ring fairly easily. As before, the differences in the stellar positions are discussed in the previous section on black holes. Comparison of the changes in the apparent positions of the surface features are quite similar but harder to discern in the figures.

The viewer is now taken even closer to the neutron star. Fig. 3k shows the distortions from 3 R_S (12.6 km): at twice the height of the photon sphere. The viewer is still looking directly at the neutron star.

Figs. 3l, 3m, and 3n show distortions from 1.5 R_S (6.3 km), the height of the photon sphere, as the viewer's inspection angle pans up. The relative angles the viewer is looking with respect the direction of the neutron star are 30°, 60°, and 90°, respectively.

In Fig. 3n, the viewer now looks directly along the photon sphere. The self Einstein ring where viewers could see the backs of their heads is the photon sphere horizon line dividing the land and the sky. In exact accordance with the black hole case (Figs. 2l, 2m, and 2n), Fig. 3n shows the first sky Einstein ring would be an invisible line about 2/9ths of the way toward the top of the plot above the photon sphere. Similarly, the first surface Einstein ring could be drawn in as a line just under (about a line's width below) the photon sphere.

The apparent color of the surface features would now appear less redshifted. This is because the light no longer has to climb out of so deep a potential well to reach the observer. The surface would still appear redshifted to some degree, as the light must climb from the surface to the photon sphere, but not nearly so much as before. The sky would appear more blueshifted than before. In fact, the sky would appear to have the same appearance and blueshift as if the observer was the same distance from a black hole.

The viewer now does an orbit at the height of the photon sphere, as shown in Figs. 3n - 3s. Fig. 3o shows the view 5° from the Fig. 3n into the orbit, while Figs. 3p, 3q, 3r, and 3s are 90°, 180°, 270°, and 360° respectively. As the viewer moves along the photon sphere, both the stellar images and the surface images appear to move in peculiar ways. Stars that would have appeared behind the observer in the absence of large gravitational light deflection effects now have images that appear in front of the viewer and above the photon sphere, but below the first sky Einstein ring. A star that approaches the opposite side of the neutron star from the viewer appears to approach the first Einstein ring from below, as this secondary image get brighter. As the motion of the viewer causes the star to move relatively from behind to in front of the viewer (crossing close to the opposite side of the neutron star), the secondary stellar image moves much faster, reaches it maximum brightness, and moves quickly (below the first sky Einstein ring) out of the picture. Shortly thereafter, the brighter primary stellar image above the first sky Einstein ring comes into view on the opposite side of the plot. As the viewer moves further around in its orbit, this primary stellar image rises and dims.

Similarly, surface features actually just behind the viewer are visible well in front of the viewer just below the photon sphere but above the first surface Einstein ring.

The viewer now descends to the neutron star surface. This is shown in Figs. 3t and 3u. Fig. 3t shows the surface distortions from a height of 1.4 R_S (5.88 km) and Fig. 3u illustrates the distortions visible from just above the surface, at a height of 1.33 R_S (5.6 km).

The viewer now appears to be in a slight bowl. Looking horizontally the viewer does not see the sky but rather the surface. Even looking at an angle slightly up, the viewer would be observing the surface, no matter which azimuthal direction is observed. This is because of the photon orbits that are trapped. Some photon orbits leave the neutron star and fall back, never reaching infinity. If the viewer looks along these paths, the viewer will be looking slightly up and still seeing an image of the surface. In fact the sky now appears scrunched up to occupy a smaller "hole" above the observer. Now the surface appears to have no redshift at all, while the sky has its maximum blueshift.

As before, the apparent line dividing land and sky still marks the apparent position of the photon sphere. Also, as before, when orbiting at the photon sphere, the sky Einstein rings are all seen above the apparent photon sphere position. The first sky Einstein ring is about 1/9 of the length of the plot border above the photon sphere. The many surface Einstein rings and complete images of the stellar surface are contained just below the photon sphere, although they together occupy only a thin sliver below the photon sphere.

In the last sequence, the viewer pans around to see the whole neutron star surface. This is shown in Figs. 3v - 3w, showing the surface image distortions at relative viewing angles of 120°, and 240°. The entire surface and sky is visible, but not in a single field of view. Lastly, Fig. 3x has the viewer looking along the original northward direction, the same as Fig. 3u.

V. COMMENTS

An interesting coincidence is that the size of the "ultracompact neutron star earth" pictured in Figs. 1d, 2d, and 3d is only a factor of a four smaller than the size the earth would be were it actually compacted to neutron star or black hole densities. Were the earth compacted to these densities, however, it would explode. The earth does not have the gravitational mass needed to suppress the enormous repulsive force between nuclei at densities this high.

The type of map projections the Schwarzschild metric creates are different than any commonly used type of map projection,[33] and different than any known type of map projection that the author is aware of.

A video has been produced featuring these lens effects[34] and is available free of charge. To receive a copy of the VHS tape, please write to the author's address given under the paper's title. The author will try to maintain an abundance of copies of the video on the latest popular video display medium (be it HDTV tape, laser disc, etc.) to service requests even several years after this article's publication.

ACKNOWLEDGEMENTS

I would like to thank Christ Ftaclas and Kent Wood for initially stimulating interest in the project and for helpful suggestions. Additionally I would like to thank Peter Becker, Peter Noerdlinger, Kevin Rauch, Brad Stuart, Mark Stuckey, John Wallin, and Daryl Yentis for advice and helpful discussions. This paper is dedicated to the memory of Ana Nash.

APPENDIX

What is the apparent angular size of a sphere of mass M (corresponding to Schwarzschild radius R_S), radius R_*, at distance D? This extremely simple and beautiful problem is important to observable aspects of high gravity environments.

The impact parameter at infinity, here labeled b, is a constant along the trajectory of the photon.[28] At any radius r from a sphere with mass M with Schwarzschild R_S on the photon's trajectory, this constant is equal to

$$b = \frac{r \sin\delta}{\sqrt{1 - R_S/r}}, \tag{A1}$$

where δ is the angle the photon's velocity makes with the radial direction.

For a lens large enough not to exhibit a photon sphere ($R_* > 1.5R_S$), the limb of the source will be seen when a photon leaving the source tangentially grazing its surface, such that $r = R_*$ and $\delta = \pi/2$. Then

$$b = \frac{R_*}{\sqrt{1 - R_S/R_*}}. \tag{A2}$$

This photon will reach the observer with $r = D$ and angle to the center of the lens of δ_{obs} such that

$$b = \frac{D \sin\delta_{obs}}{\sqrt{1 - R_S/D}}. \tag{A3}$$

Equating Eqs. (A2) and (A3) and solving for δ_{obs} yields

$$\delta_{obs} = \arcsin\left(\frac{R_*\sqrt{1 - R_S/D}}{D\sqrt{1 - R_S/R_*}}\right), \tag{A4}$$

which is the apparent size of the lens.

When the lens is compact enough to be surrounded by a photon sphere, one can simply replace R_* in the above expression with $1.5R_S$, as these photons now define the apparent size of the lens. The result is

$$\delta_{obs} = \arcsin\left(\frac{3\sqrt{3}R_S\sqrt{1-R_S/D}}{2D}\right). \tag{A5}$$

Be careful to assign the correct quadrant to the result. If $R_* < 1.5R_S$, then δ_{obs} will be greater than $\pi/2$.

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FIGURE CAPTIONS

Fig. 1: Visual distortions near a "normal" neutron star. The neutron star depicted has neither a photon sphere nor an event horizon.

Fig. 2: Visual distortions near a black hole. A black hole has both a photon sphere and an event horizon.

Fig. 3: Visual distortions near an ultracompact neutron star. This hypothetical neutron star has a photon sphere but no event horizon.